# Notes on 2+1D Space Charge

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A useful formula is that if the full charge density is separable

$$\rho_{\mathrm{3D}} = \rho(x,y)\lambda(z),$$

then the space charge equation $\nabla^2\phi = \rho_{\mathrm{3D}}$ can be solved exactly by the infinite sum

$$\begin{aligned}\phi &= \sum_{n=0}^{\infty}(-1)^n\ \nabla^{-2(n+1)}\rho\ \lambda^{(2n)} \\ &= \sum_{n=0}^{\infty}(-1)^n\ \phi_{2D,n}\ \lambda^{(2n)},\end{aligned}$$

where we may calculate $\nabla^2\phi_{2D,0} = \rho$ and $\nabla^2\phi_{2D,n+1} = \phi_{2D,n}$ via Poisson solvers. The associated force law is

$$\boldsymbol{F} = \boldsymbol{\nabla}\phi = \sum_{n=0}^{\infty}(-1)^n \begin{bmatrix} \boldsymbol{\nabla}\phi_{2D,n}\ \lambda^{(2n)} \\ \phi_{2D,n}\ \lambda^{(2n+1)} \end{bmatrix}.$$

This series converges provided the repeated integrations transversely more than cancel the repeated derivatives longitudinally, which happens if $\rho$ varies spatially faster than $\lambda$.

It is also true that a non-separable density may always be expanded into a sum of separable terms

$$\rho_{\mathrm{3D}} = \sum_{n=0}^{\infty}\rho_n(x,y)\lambda_n(z),$$

each of which may be treated by the previous series and linearly combined. The convergence of this sum depends on the complexity of the charge distribution (the degree of longitudinal-transverse coupling), whereas the convergence of the other depends on the aspect ratio of detail in a long beam.

The traditional 2+1D formula corresponds in part to the $n = 0$ term of both series. Its force is proportional to

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{\nabla}\phi_{2D,0}\ \lambda \\ \phi_{2D,0}\ \lambda' \end{bmatrix}.$$

$\phi_{2D,0}$ is the solution to the usual transverse-only space charge problem. The transverse force term is exactly as expected, but the longitudinal term has $\lambda'$ multiplied by the potential instead of a constant. Which other terms in the expansion make this resemble a constant times $\lambda'$ is unclear, in fact it is not clear that a constant times $\lambda'$ is a better approximation than this first term.