A useful formula is that if the full charge density is separable
\[ \rho_{3D} = \rho(x, y) \lambda(z), \]
then the space charge equation \( \nabla^2 \phi = \rho_{3D} \) can be solved exactly by the infinite sum
\[
\phi = \sum_{n=0}^{\infty} (-1)^n \nabla^{-2(n+1)} \rho \lambda^{(2n)} = \sum_{n=0}^{\infty} (-1)^n \phi_{2D,n} \lambda^{(2n)},
\]
where we may calculate \( \nabla^2 \phi_{2D,0} = \rho \) and \( \nabla^2 \phi_{2D,n+1} = \phi_{2D,n} \) via Poisson solvers. The associated force law is
\[
F = \nabla \phi = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{\nabla \phi_{2D,n}}{\phi_{2D,n}} \lambda^{(2n)} \frac{\lambda}{\lambda^{(2n+1)}} \right].
\]
This series converges provided the repeated integrations transversely more than cancel the repeated derivatives longitudinally, which happens if \( \rho \) varies spatially faster than \( \lambda \).

It is also true that a non-separable density may always be expanded into a sum of separable terms
\[ \rho_{3D} = \sum_{n=0}^{\infty} \rho_n(x, y) \lambda_n(z), \]
each of which may be treated by the previous series and linearly combined. The convergence of this sum depends on the complexity of the charge distribution (the degree of longitudinal-transverse coupling), whereas the convergence of the other depends on the aspect ratio of detail in a long beam.

The traditional 2+1D formula corresponds in part to the \( n = 0 \) term of both series. Its force is proportional to
\[
F = \begin{bmatrix} \nabla \phi_{2D,0} \lambda \\ \phi_{2D,0} \lambda' \end{bmatrix}.
\]
\( \phi_{2D,0} \) is the solution to the usual transverse-only space charge problem. The transverse force term is exactly as expected, but the longitudinal term has \( \lambda' \) multiplied by the potential instead of a constant. Which other terms in the expansion make this resemble a constant times \( \lambda' \) is unclear, in fact it is not clear that a constant times \( \lambda' \) is a better approximation than this first term.