Fringe Fields for VFFAG Magnets with Edge Angles

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1 Conventions, Assumptions and Definitions

The beam is travelling in the $+z$ direction and the machine closed orbit shifts in the $+y$ direction (‘upwards’) with increasing energy. The standard hard-edged scaling VFFAG field is $B_y = B_0 e^{ky}$ within the magnet body on the $x = 0$ mid-plane [1], where $k$ has units of inverse length.

For a skew magnet, the edge angle is parametrised by $\tau = \tan \theta$, where $\theta$ is the angle between the magnet edge and the $y$ axis. Defining $\zeta = z - \tau y$, the function $f(\zeta)$ determines the field fall off at both ends of the magnet, approaching $f = 1$ within the magnet and $f = 0$ outside.

2 Fringe Field Issues

A standard Maxwellian extrapolation [2] away from the $x = 0$ plane is insufficient because it only satisfies the three Maxwell equations involving $\partial_x$ and there is also the curl condition $(\nabla \times \mathbf{B})_x = \partial_y B_z - \partial_z B_y = 0$ to satisfy within the plane itself. This is not a problem for conventional dipoles where only $B_x$ (the perpendicular field component) is nonzero on the midplane.

2.1 A General Solution for Zero Edge Angle

Given a desired $B_y$, the curl condition can be enforced explicitly via

$$\partial_y B_z - \partial_z B_y = 0 \quad \Rightarrow \quad \partial_y B_z = \partial_z B_y \quad \Rightarrow \quad B_z = \int^y \partial_z B_y \, dy,$$

where the lower $y$-limit of the integral can be chosen as desired for each value of $z$. If the goal is to have $B_y = f(z)$, the field components

$$B_y = f(z), \quad B_z = y f'(z)$$

satisfy $\partial_y B_z = \partial_z B_y = f'(z)$. For the more general situation $B_y = g(y, z)$, the field

$$B_y = g(y, z), \quad B_z = \int^y \partial_z g(y, z) \, dy$$

satisfies $\partial_y B_z = \partial_z B_y = \partial_z g(y, z)$.
2.2 Extension to Angled Fringe Fields

Although the solution in the previous section can be used with \( g(y, z) \) equal to the desired slanted field \( f(\zeta) \), this produces fringe fields that propagate vertically rather than parallel to the magnet edge. However, consider the field

\[ B_y = f(\zeta) - \tau y f'(\zeta), \quad B_z = y f'(\zeta). \]

Since \( \partial_y f(\zeta) = -\tau f'(\zeta) \) and \( \partial_z f(\zeta) = f'(\zeta) \), the curl condition is satisfied with \( \partial_y B_z = \partial_z B_y = f'(\zeta) - \tau y f''(\zeta) \). Can this be generalised in an analogous way to the zero edge angle case, by replacing a multiplication by \( y \) with an integral in \( y \)? For a function \( g(y, \zeta) \) this would suggest the field

\[ B_y = g(y, \zeta) - \tau \int_y^\infty \partial_\zeta g(y, \zeta) \, dy, \quad B_z = \int_y^\infty \partial_\zeta g(y, \zeta) \, dy. \]

Checking the derivatives gives

\[ \partial_y B_z = \partial_z B_y = \partial_\zeta g(y, \zeta) - \tau \int_y^\infty \partial_\zeta^2 g(y, \zeta) \, dy, \]

noting firstly that \( \zeta \) depends on \( y \), so \( g(y, \zeta) \) depends on \( y \) twice and secondly that \( \partial_z = \partial_\zeta \) since \( y \) is held constant in both cases. The lower limit of the \( B_z \) integral is still free to be chosen, this time as a function of \( \zeta \) and its evaluation can be reused in \( B_y = g(y, \zeta) - \tau B_z \).

2.3 Scalar Potential Formulation

The fringe field issue is caused by having to satisfy \( \nabla \times \mathbf{B} = 0 \) in free space. However if \( \mathbf{B} = \nabla \phi \), the curl of a gradient is zero so this equation always holds. The four fields given in this section have the scalar potentials below (valid only on the \( x = 0 \) plane).

\[
\begin{align*}
\phi_{f0} &= y f(z) \\
\phi_{g0} &= \int_y^\infty g(y, z) \, dy \\
\phi_{f\theta} &= y f(\zeta) \\
\phi_{g\theta} &= \int_y^\infty g(y, \zeta) \, dy
\end{align*}
\]

3 The Scaling VFFAG Mid-Plane Field

Using the angled fringe field formula with \( g(y, \zeta) = B_0 e^{ky} f(\zeta) \) yields

\[ B_z = \int_y^\infty B_0 e^{ky} f'(\zeta) \, dy = B_0 f'(\zeta) \frac{e^{ky} - e^{kY(\zeta)}}{k}, \]

where \( Y(\zeta) \) is the lower limit of the integral. Because the integral is bounded in the negative \( y \) direction (assuming \( k > 0 \)), the most elegant choice here is \( Y(\zeta) = -\infty \), so that

\[ B_z = B_0 e^{ky} \frac{f'(\zeta)}{k}, \quad B_y = B_0 e^{ky} f(\zeta) - \tau B_z = B_0 e^{ky} \left( f(\zeta) - \frac{\tau f'(\zeta)}{k} \right). \]
4 Off-plane Field Expansion

This section adapts the magnetic field expansion used in [2] for the vertical orbit-excursion case. Maxwell’s equations in free space, $\nabla \cdot B = \nabla \times B = 0$, can be rearranged to give

$$\partial_x B = \begin{bmatrix} 0 & -\partial_y & -\partial_z \\ \partial_y & 0 & 0 \\ \partial_z & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & -\nabla_{y,z}^T \\ \nabla_{y,z} & 0 \end{bmatrix} B,$$

from which a Taylor expansion in $x$ yields

$$B_x(x, y, z) = \sum_{n=0}^{\infty} x^{2n+1} \frac{1}{(2n+1)!} (-\nabla_{y,z}^2)^n (-\nabla_{y,z} \cdot B_{y,z}(0, y, z))$$

$$B_{y,z}(x, y, z) = B_{y,z}(0, y, z) + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \nabla_{y,z} (-\nabla_{y,z}^2)^{n-1} (-\nabla_{y,z} \cdot B_{y,z}(0, y, z)).$$

Here it has been assumed that $B_x(0, y, z) = 0$. Since our field has a scalar potential on the midplane, $B_{y,z}(0, y, z) = \nabla_{y,z} \phi(y, z)$, which simplifies the formulae further:

$$B_x(x, y, z) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-\nabla_{y,z}^2)^{n+1} \phi(y, z)$$

$$B_{y,z}(x, y, z) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \nabla_{y,z} (-\nabla_{y,z}^2)^n \phi(y, z).$$

For the VFFAG mid-plane field, $\phi(y, z) = B_0 e^{k_y f(\zeta)}$.

4.1 The Iterated 2D Laplacian

The bulk of the field calculation is now in evaluating $(-\nabla_{y,z}^2)^n \phi(y, z)$ from the summands in the previous section. Assuming $f$ and its derivatives are evaluated at $\zeta$ unless otherwise stated, it is useful to evaluate

$$-\nabla_{y,z}^2 (e^{k_y f(n)}) = -\partial_y \partial_y (e^{k_y f(n)}) - \partial_z \partial_z (e^{k_y f(n)})$$

$$= -\partial_y (ke^{k_y f(n)} - \tau e^{k_y f(n+1)}) - \partial_z (e^{k_y f(n+1)})$$

$$= -(k^2 e^{k_y f(n)} - 2k\tau e^{k_y f(n+1)} + \tau^2 e^{k_y f(n+2)}) - e^{k_y f(n+2)}$$

Therefore the general form is

$$(-\nabla_{y,z}^2)^n \phi(y, z) = B_0 e^{k_y} \sum_{j=0}^{2n} a_{n,j} f^{(j)}$$

and the recurrence relation is

$$a_{n+1,j} = -k^2 a_{n,j} + 2k\tau a_{n,j-1} - (1 + \tau^2) a_{n,j-2},$$

where the definition is made that $a_{n,-1} = a_{n,-2} = 0$. Initially, $a_{00} = \frac{1}{k}$ and $a_{0j} = 0$ elsewhere.
4.2 Field Evaluation Formulae for General $f(\zeta)$

Noting that
\[ \nabla_{y,z}(-\nabla_{y,z}^2)^n \phi(y, z) = B_0 e^{ky} \sum_{j=0}^{2n} a_{nj} \left[ k f^{(j)} - \tau f^{(j+1)} \right], \]
the field formulae become
\[ B_x(x, y, z) = B_0 e^{ky} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \sum_{j=0}^{2n+2} a_{n+1,j} f^{(j)}, \]
\[ B_{y,z}(x, y, z) = B_0 e^{ky} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \sum_{j=0}^{2n} a_{nj} \left[ k f^{(j)} - \tau f^{(j+1)} \right]. \]

Here, $a_{nj}$ can be precalculated for each magnet (specific $k$, $\tau$ values) but the derivatives $f^{(j)}$ need to be found for each point ($\zeta$ value). After the scaling factor $B_0 e^{ky}$, the rest of the formula is a function of $x$ and $\zeta$ only, so it suffices to produce a 2D field map and use the scaling law
\[ B(x, y, z) = e^{ky} B(x, 0, z - \tau y). \]

5 tanh Fringe Field Derivatives

Given a fringe field extent $l$, the field fall off function can have the form
\[ f(\zeta) = \sigma(\zeta/l) - \sigma((\zeta - L)/l), \]
where $\sigma(x)$ is a unit sigmoid function going from 0 to 1. The $n^{th}$ derivative is
\[ f^{(n)}(\zeta) = l^{-n} \left( \sigma^{(n)}(\zeta/l) - \sigma^{(n)}((\zeta - L)/l) \right). \]

A common sigmoid function with exponential fall off characteristics (e.g. for iron fringe field shielding) is $\sigma(x) = \frac{1}{2} + \frac{1}{2} \tanh x$. With this choice of $\sigma(x)$, the constant offset of $\frac{1}{2}$ cancels in the expression for $f$:
\[ f^{(n)}(\zeta) = \frac{l^{-n}}{2} \left( \tanh^{(n)}(\zeta/l) - \tanh^{(n)}((\zeta - L)/l) \right). \]

5.1 Computational Simplification

Usefully, tanh satisfies the relation
\[ \frac{d \tanh x}{dx} = 1 - \tanh^2 x \quad \Rightarrow \quad \frac{d \tanh^n x}{dx} = n \tanh^{n-1} x (1 - \tanh^2 x) = n \tanh^{n-1} x - n \tanh^{n+1} x, \]
which can be used to generate a recurrence for $\tanh^{(n)} x$ in terms of powers of $\tanh x$:
\[ \tanh^{(n)} x = \sum_{j=0}^{n+1} t_{nj} \tanh^j x \]
\[ t_{01} = 1, \quad t_{0j} = 0 \quad (j \neq 1), \quad t_{n+1,j} = (j+1) t_{n,j+1} - (j-1) t_{n,j-1}. \]

Substituting this into the formula for $f^{(n)}$ gives
\[ f^{(n)}(\zeta) = \frac{l^{-n}}{2} \sum_{j=0}^{n+1} t_{nj} \left( \tanh^j(\zeta/l) - \tanh^j((\zeta - L)/l) \right) = \frac{l^{-n}}{2} \sum_{j=0}^{n+1} t_{nj}(T_1^j - T_2^j), \]
where $T_1 = \tanh(\zeta/l)$ and $T_2 = \tanh((\zeta - L)/l)$ can be precalculated for each $\zeta$. 

5.2 Calculating the Field Map

Precalculate $t_{n_j}$ to the desired order and $a_{n_j}$ for this specific magnet. For each value of $\zeta$ calculate $T_1$ and $T_2$, then the vector of $T_1^n - T_2^n$ values that gives the vector of $f^{(n)}$ after a linear transformation using $t_{n_j}$. Calculate two further linear transformations using $a_{n_j}$:

$$g_n = \sum_{j=0}^{2n} a_{n_j} f^{(j)} \quad \text{and} \quad h_n = \sum_{j=0}^{2n} a_{n_j} f^{(j+1)}.$$

Assuming the field map is being made at $y=0$, for each $x$ value, the field components are:

$$B_x(x, 0, z) = B_0 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} g_{n+1}$$

$$B_{y,z}(x, 0, z) = B_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \begin{bmatrix} kg_n - \tau h_n \\ h_n \end{bmatrix}.$$

References
