Fringe Fields for VFFAG Magnets with Edge Angles

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1 Conventions, Assumptions and Definitions

The beam is travelling in the $+z$ direction and the machine closed orbit shifts in the $+y$ direction ('upwards') with increasing energy. The standard hard-edged scaling VFFAG field is $B_y = B_0 \mathrm{e}^{ky}$ within the magnet body on the $x = 0$ mid-plane [1], where k has units of inverse length.

For a skew magnet, the edge angle is parametrised by $\tau = \tan\theta$, where θ is the angle between the magnet edge and the y axis. Defining $\zeta = z - \tau y$, the function $f(\zeta)$ determines the field fall off at both ends of the magnet, approaching $f = 1$ within the magnet and $f = 0$ outside.

2 Fringe Field Issues

A standard Maxwellian extrapolation [2] away from the $x = 0$ plane is insufficient because it only satisfies the three Maxwell equations involving ∂_x and there is also the curl condition $(\nabla \times \mathbf{B})_x = \partial_y B_z - \partial_z B_y = 0$ to satisfy within the plane itself. This is not a problem for conventional dipoles where only B_x (the perpendicular field component) is nonzero on the midplane.

2.1 A General Solution for Zero Edge Angle

Given a desired B_y, the curl condition can be enforced explicitly via

$$\partial_y B_z - \partial_z B_y = 0 \qquad \Rightarrow \qquad \partial_y B_z = \partial_z B_y \qquad \Rightarrow \qquad B_z = \int^y \partial_z B_y \,\mathrm{d}y,$$

where the lower y-limit of the integral can be chosen as desired for each value of z. If the goal is to have $B_y = f(z)$, the field components

$$B_y = f(z), \qquad B_z = y f'(z)$$

satsify $\partial_y B_z = \partial_z B_y = f'(z)$. For the more general situation $B_y = g(y, z)$, the field

$$B_y = g(y, z), \qquad B_z = \int^y \partial_z g(y, z) \,\mathrm{d}y$$

satisfies $\partial_y B_z = \partial_z B_y = \partial_z g(y, z)$.

2.2 Extension to Angled Fringe Fields

Although the solution in the previous section can be used with $g(y,z)$ equal to the desired slanted field $f(\zeta)$, this produces fringe fields that propagate vertically rather than parallel to the magnet edge. However, consider the field

$$B_y = f(\zeta) - \tau y f'(\zeta), \qquad B_z = y f'(\zeta).$$

Since $\partial_y f(\zeta) = -\tau f'(\zeta)$ and $\partial_z f(\zeta) = f'(\zeta)$, the curl condition is satisfied with $\partial_y B_z = \partial_z B_y = f'(\zeta) - \tau y f''(\zeta)$. Can this be generalised in an analogous way to the zero edge angle case, by replacing a multiplication by y with an integral in y? For a function $g(y,\zeta)$ this would suggest the field

$$B_y = g(y,\zeta) - \tau \int^y \partial_\zeta g(y,\zeta)\,\mathrm{d}y, \qquad B_z = \int^y \partial_\zeta g(y,\zeta)\,\mathrm{d}y.$$

Checking the derivatives gives

$$\partial_y B_z = \partial_z B_y = \partial_\zeta g(y,\zeta) - \tau \int^y \partial_\zeta^2 g(y,\zeta)\,\mathrm{d}y,$$

noting firstly that ζ depends on y, so $g(y,\zeta)$ depends on y twice and secondly that $\partial_z = \partial_\zeta$ since y is held constant in both cases. The lower limit of the B_z integral is still free to be chosen, this time as a function of ζ and its evaluation can be reused in $B_y = g(y,\zeta) - \tau B_z$.

2.3 Scalar Potential Formulation

The fringe field issue is caused by having to satisfy $\nabla \times \mathbf{B} = \mathbf{0}$ in free space. However if $\mathbf{B} = \nabla\phi$, the curl of a gradient is zero so this equation always holds. The four fields given in this section have the scalar potentials below (valid only on the $x = 0$ plane).

$$\begin{aligned}
\phi_{f0} &= y f(z) \\
\phi_{g0} &= \int^y g(y,z)\,\mathrm{d}y \\
\phi_{f\theta} &= y f(\zeta) \\
\phi_{g\theta} &= \int^y g(y,\zeta)\,\mathrm{d}y
\end{aligned}$$

3 The Scaling VFFAG Mid-Plane Field

Using the angled fringe field formula with $g(y,\zeta) = B_0 \mathrm{e}^{ky} f(\zeta)$ yields

$$B_z = \int^y B_0 \mathrm{e}^{ky} f'(\zeta)\,\mathrm{d}y = B_0 f'(\zeta)\,\frac{\mathrm{e}^{ky} - \mathrm{e}^{kY(\zeta)}}{k},$$

where $Y(\zeta)$ is the lower limit of the integral. Because the integral is bounded in the negative y direction (assuming $k > 0$), the most elegant choice here is $Y(\zeta) = -\infty$, so that

$$B_z = B_0 \mathrm{e}^{ky} \frac{f'(\zeta)}{k}, \qquad B_y = B_0 \mathrm{e}^{ky} f(\zeta) - \tau B_z = B_0 \mathrm{e}^{ky} \left(f(\zeta) - \frac{\tau f'(\zeta)}{k} \right).$$

4 Off-plane Field Expansion

This section adapts the magnetic field expansion used in [2] for the vertical orbit-excursion case. Maxwell's equations in free space, $\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$, can be rearranged to give

$$\partial_x \mathbf{B} = \begin{bmatrix} 0 & -\partial_y & -\partial_z \\ \partial_y & 0 & 0 \\ \partial_z & 0 & 0 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0 & -\nabla_{y,z}^T \\ \nabla_{y,z} & 0 \end{bmatrix} \mathbf{B},$$

from which a Taylor expansion in x yields

$$B_x(x,y,z) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-\nabla_{y,z}^2)^n (-\nabla_{y,z} \cdot \mathbf{B}_{y,z}(0,y,z))$$

$$\mathbf{B}_{y,z}(x,y,z) = \mathbf{B}_{y,z}(0,y,z) + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \nabla_{y,z} (-\nabla_{y,z}^2)^{n-1} (-\nabla_{y,z} \cdot \mathbf{B}_{y,z}(0,y,z)).$$

Here it has been assumed that $B_x(0,y,z) = 0$. Since our field has a scalar potential on the midplane, $\mathbf{B}_{y,z}(0,y,z) = \nabla_{y,z}\phi(y,z)$, which simplifies the formulae further:

$$B_x(x,y,z) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-\nabla_{y,z}^2)^{n+1} \phi(y,z)$$

$$\mathbf{B}_{y,z}(x,y,z) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \nabla_{y,z} (-\nabla_{y,z}^2)^n \phi(y,z).$$

For the VFFAG mid-plane field, $\phi(y,z) = B_0 \mathrm{e}^{ky} \frac{f(\zeta)}{k}$.

4.1 The Iterated 2D Laplacian

The bulk of the field calculation is now in evaluating $(-\nabla_{y,z}^2)^n \phi(y,z)$ from the summands in the previous section. Assuming f and its derivatives are evaluated at ζ unless otherwise stated, it is useful to evaluate

$$\begin{aligned} -\nabla_{y,z}^2(\mathrm{e}^{ky} f^{(n)}) &= -\partial_y\partial_y(\mathrm{e}^{ky} f^{(n)}) - \partial_z\partial_z(\mathrm{e}^{ky} f^{(n)}) \\ &= -\partial_y(k\mathrm{e}^{ky} f^{(n)} - \tau \mathrm{e}^{ky} f^{(n+1)}) - \partial_z(\mathrm{e}^{ky} f^{(n+1)}) \\ &= -(k^2 \mathrm{e}^{ky} f^{(n)} - 2k\tau \mathrm{e}^{ky} f^{(n+1)} + \tau^2 \mathrm{e}^{ky} f^{(n+2)}) - \mathrm{e}^{ky} f^{(n+2)} \\ &= \mathrm{e}^{ky}(-k^2 f^{(n)} + 2k\tau f^{(n+1)} - (1+\tau^2) f^{(n+2)}). \end{aligned}$$

Therefore the general form is

$$(-\nabla_{y,z}^2)^n \phi(y,z) = B_0 \mathrm{e}^{ky} \sum_{j=0}^{2n} a_{nj} f^{(j)}$$

and the recurrence relation is

$$a_{n+1,j} = -k^2 a_{nj} + 2k\tau a_{n,j-1} - (1+\tau^2) a_{n,j-2},$$

where the definition is made that $a_{n,-1} = a_{n,-2} = 0$. Initially, $a_{00} = \frac{1}{k}$ and $a_{0j} = 0$ elsewhere.

4.2 Field Evaluation Formulae for General $f(\zeta)$

Noting that

$$\nabla_{y,z}(-\nabla^2_{y,z})^n\phi(y,z) = B_0 \mathrm{e}^{ky}\sum_{j=0}^{2n} a_{nj}\begin{bmatrix} kf^{(j)} - \tau f^{(j+1)} \\ f^{(j+1)} \end{bmatrix},$$

the field formulae become

$$B_x(x,y,z) = B_0\mathrm{e}^{ky}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\sum_{j=0}^{2n+2} a_{n+1,j}f^{(j)},$$

$$\mathbf{B}_{y,z}(x,y,z) = B_0\mathrm{e}^{ky}\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\sum_{j=0}^{2n} a_{nj}\begin{bmatrix} kf^{(j)} - \tau f^{(j+1)} \\ f^{(j+1)} \end{bmatrix}.$$

Here, a_{nj} can be precalculated for each magnet (specific k, τ values) but the derivatives $f^{(j)}$ need to be found for each point (ζ value). After the scaling factor $B_0\mathrm{e}^{ky}$, the rest of the formula is a function of x and ζ only, so it suffices to produce a 2D field map and use the scaling law

$$\mathbf{B}(x,y,z) = \mathrm{e}^{ky}\ \mathbf{B}(x,0,z-\tau y).$$

5 tanh Fringe Field Derivatives

Given a fringe field extent l, the field fall off function can have the form

$$f(\zeta) = \sigma(\zeta/l) - \sigma((\zeta - L)/l),$$

where $\sigma(x)$ is a unit sigmoid function going from 0 to 1. The n^{th} derivative is

$$f^{(n)}(\zeta) = l^{-n}\left(\sigma^{(n)}(\zeta/l) - \sigma^{(n)}((\zeta-L)/l)\right).$$

A common sigmoid function with exponential fall off characteristics (e.g. for iron fringe field shielding) is $\sigma(x) = \frac{1}{2} + \frac{1}{2}\tanh x$. With this choice of $\sigma(x)$, the constant offset of $\frac{1}{2}$ cancels in the expression for f:

$$f^{(n)}(\zeta) = \frac{l^{-n}}{2}\left(\tanh^{(n)}(\zeta/l) - \tanh^{(n)}((\zeta-L)/l)\right).$$

5.1 Computational Simplification

Usefully, tanh satisfies the relation

$$\frac{\mathrm{d}\tanh x}{\mathrm{d}x} = 1-\tanh^2 x \quad \Rightarrow \quad \frac{\mathrm{d}\tanh^n x}{\mathrm{d}x} = n\tanh^{n-1}x(1-\tanh^2 x) = n\tanh^{n-1}x - n\tanh^{n+1}x,$$

which can be used to generate a recurrence for $\tanh^{(n)} x$ in terms of powers of $\tanh x$:

$$\tanh^{(n)} x = \sum_{j=0}^{n+1} t_{nj}\tanh^j x$$

$$t_{01} = 1, \quad t_{0j} = 0 \quad (j \neq 1), \qquad t_{n+1,j} = (j+1)t_{n,j+1} - (j-1)t_{n,j-1}.$$

Substituting this into the formula for $f^{(n)}$ gives

$$f^{(n)}(\zeta) = \frac{l^{-n}}{2}\sum_{j=0}^{n+1} t_{nj}\left(\tanh^j(\zeta/l) - \tanh^j((\zeta-L)/l)\right) = \frac{l^{-n}}{2}\sum_{j=0}^{n+1} t_{nj}(T_1^j - T_2^j),$$

where $T_1 = \tanh(\zeta/l)$ and $T_2 = \tanh((\zeta-L)/l)$ can be precalculated for each ζ.

5.2 Calculating the Field Map

Precalculate t_{nj} to the desired order and a_{nj} for this specific magnet. For each value of ζ calculate T_1 and T_2, then the vector of $T_1^n - T_2^n$ values that gives the vector of $f^{(n)}$ after a linear transformation using t_{nj}. Calculate two further linear transformations using a_{nj}:

$$g_n = \sum_{j=0}^{2n} a_{nj} f^{(j)} \qquad \text{and} \qquad h_n = \sum_{j=0}^{2n} a_{nj} f^{(j+1)}.$$

Assuming the field map is being made at $y = 0$, for each x value, the field components are:

$$B_x(x, 0, z) = B_0 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\, g_{n+1}$$

$$\mathbf{B}_{y,z}(x, 0, z) = B_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \begin{bmatrix} kg_n - \tau h_n \\ h_n \end{bmatrix}.$$

References

[1] *Vertical Orbit Excursion FFAGs*, S.J. Brooks, Proc. HB2010.

[2] *Extending the Energy Range of 50 Hz Proton FFAGs*, S.J. Brooks, Proc. PAC 2009.