Magnet Field of a Finite Wire

Stephen Brooks

October 24, 2011

1 Assumptions

The wire segment in question travels in a straight line from position $\mathbf{a}$ to $\mathbf{b}$ and carries current I in the direction towards $\mathbf{b}$. The magnetic field of this segment alone will not be Maxwellian because the current does not satisfy the continuity equation. However, the field sum of a loop of such wires will be.

2 Derivation

The Biot-Savart law is

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \mathbf{r}}{|\mathbf{r}|^3} \, \mathrm{d}s,$$

where $\mathbf{r}$ is the vector to $\mathbf{x}$ from the relevant point on the conductor. In this case the parametrisation

$$\mathbf{r} = \mathbf{x} - (\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})), \qquad \mathrm{d}s = |\mathbf{b} - \mathbf{a}| \, \mathrm{d}\lambda, \qquad \mathbf{I} = \frac{I(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|},$$

for $\lambda \in [0, 1]$, is used. Now

$$\mathbf{I} \times \mathbf{r} = \frac{I(\mathbf{b} - \mathbf{a}) \times (\mathbf{x} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|} \qquad \Rightarrow \qquad \mathbf{I} \times \mathbf{r} \, \mathrm{d}s = I(\mathbf{b} - \mathbf{a}) \times (\mathbf{x} - \mathbf{a}) \, \mathrm{d}\lambda$$

$$\Rightarrow \qquad \mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} (\mathbf{b} - \mathbf{a}) \times (\mathbf{x} - \mathbf{a}) \int_0^1 \frac{\mathrm{d}\lambda}{|\mathbf{r}|^3}.$$

The length of $\mathbf{r}$ can be calculated via

$$|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = |\mathbf{x} - \mathbf{a}|^2 - 2\lambda(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda^2 \, |\mathbf{b} - \mathbf{a}|^2$$

and the scalar integral from

$$\int_0^1 \frac{\mathrm{d}x}{(ax^2 + bx + c)^{3/2}} = \left[\frac{4ax + 2b}{(4ac - b^2)\sqrt{ax^2 + bx + c}} \right]_0^{x=1} = \frac{2}{4ac - b^2} \left(\frac{2a + b}{\sqrt{a + b + c}} - \frac{b}{\sqrt{c}} \right).$$

3 Curved Coil Elements

Suppose a coil piece is shaped as the image of a function $\mathbf{f}(u, v, w)$ applied to $(u, v, w) \in [0, 1]^3$. The total current in the coil is I and is distributed so that $I\,\mathrm{d}u\,\mathrm{d}v$ flows in the piece that is the image of $[u, u + \mathrm{d}u] \times [v, v + \mathrm{d}v] \times [0, 1]$, in the direction of 'increasing w'. The single Biot-Savart integral is

$$\mathrm{d}^2\mathbf{B}_{\text{piece}}(\mathbf{x}; u, v) = \frac{\mu_0}{4\pi} \int_{w=0}^{1} \frac{\mathrm{d}^2\mathbf{I}_{\text{piece}} \times \mathbf{r}}{|\mathbf{r}|^3}\,\mathrm{d}s,$$

where $\mathrm{d}^2\mathbf{I}_{\text{piece}}\,\mathrm{d}s = I\,\mathrm{d}u\,\mathrm{d}v \frac{\partial \mathbf{f}}{\partial w}\,\mathrm{d}w$, $\mathbf{r} = \mathbf{x} - \mathbf{f}$ and $\mathbf{f}$ means $\mathbf{f}(u, v, w)$ unless otherwise stated. Thus

$$\mathrm{d}^2\mathbf{B}_{\text{piece}}(\mathbf{x}; u, v) = \frac{\mu_0 I\,\mathrm{d}u\,\mathrm{d}v}{4\pi} \int_0^1 \frac{\frac{\partial \mathbf{f}}{\partial w} \times (\mathbf{x} - \mathbf{f})}{|\mathbf{x} - \mathbf{f}|^3}\,\mathrm{d}w$$

and the total field (triple integral) is

$$\mathbf{B}(\mathbf{x}) = \int_{u=0}^{1} \int_{v=0}^{1} \mathrm{d}^2\mathbf{B}_{\text{piece}}(\mathbf{x}; u, v) = \frac{\mu_0 I}{4\pi} \int_0^1 \int_0^1 \int_0^1 \frac{\frac{\partial \mathbf{f}}{\partial w} \times (\mathbf{x} - \mathbf{f})}{|\mathbf{x} - \mathbf{f}|^3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$