Arbitrary Beams in Uniform Pipes

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1 Beam Slice Problem

Before considering the whole beam, a simpler problem is a slice (at rest) with charge density

$$\rho = \rho_{2\mathrm{D}}(x, y)\delta(z).$$

The electric field is given by a potential $\mathbf{E} = -\nabla V$ where $V = 0$ on a perfectly conducting pipe whose shape does not change in z. Since $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, the equation to solve is $\nabla^2 V = -\rho/\epsilon_0$.

For $z \neq 0$, $\rho = 0$ so $\nabla^2 V = 0$ and the problem can be restated as a 'time evolution' problem

$$\frac{\mathrm{d}^2 V}{\mathrm{d}z^2} = -\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 V}{\mathrm{d}y^2} \qquad \Rightarrow \qquad V'' = -\nabla^2_{x,y} V,$$

where prime denotes differentiation by z and $\nabla^2_{x,y}$ is the 2D Laplacian.

1.1 Eigenmode Decomposition

Suppose there is a set of 2D eigenpotentials $V_\lambda(x, y)$ for various λ such that

$$\nabla^2_{x,y} V_\lambda = \lambda V_\lambda \qquad \text{and} \qquad V_\lambda = 0 \quad \text{on pipe.}$$

Assuming the eigenpotentials span the space of all 2D potentials within the pipe, decompose the 3D potential at each z via $V = \sum_\lambda a_\lambda(z) V_\lambda$. Now,

$$V'' = \sum_\lambda a''_\lambda V_\lambda \qquad \text{and} \qquad V'' = -\nabla^2_{x,y} V = -\sum_\lambda a_\lambda \lambda V_\lambda,$$

so equating coefficients gives

$$a''_\lambda = -\lambda a_\lambda.$$

Noting that usually for these sorts of eigenmodes, $-\lambda > 0$ and that realistically the potential should be exponentially decaying rather than increasing with z, the solution is

$$a_\lambda(z) = A_\lambda \mathrm{e}^{-\sqrt{-\lambda} z},$$

for $z \geq 0$ and some constants A_λ.

1.2 Initial Conditions

When passing through $z = 0$ we expect an infinite value of V'' to occur because of the impulse nature of ρ and the x and y contributions to the Laplacian become negligible. By symmetry, $V(x, y, -z) = V(x, y, z)$ so $V'(x, y, 0^-) = -V'(x, y, 0^+)$ and the impulse contribution to $V'' = \nabla^2 V$ at $z = 0$ is:

$$2V'(x, y, 0^+)\delta(z) = -\rho/\epsilon_0 = \frac{-\rho_{2\mathrm{D}}(x, y)\delta(z)}{\epsilon_0} \qquad \Rightarrow \qquad V'(x, y, 0^+) = \frac{-\rho_{2\mathrm{D}}(x, y)}{2\epsilon_0}.$$

Note that $V' = \sum_\lambda a'_\lambda V_\lambda$ and $a'_\lambda = -\sqrt{-\lambda} A_\lambda \mathrm{e}^{-\sqrt{-\lambda} z}$, so

$$V'(z = 0^+) = \sum_\lambda -\sqrt{-\lambda} A_\lambda V_\lambda = \frac{-\rho_{2\mathrm{D}}}{2\epsilon_0}.$$

Assuming the eigenpotentials are orthogonal and taking the dot product with a particular V_λ gives

$$\sqrt{-\lambda} A_\lambda V_\lambda \cdot V_\lambda = \frac{\rho_{2\mathrm{D}} \cdot V_\lambda}{2\epsilon_0} \qquad \Rightarrow \qquad A_\lambda = \frac{\rho_{2\mathrm{D}} \cdot V_\lambda}{2\epsilon_0 \sqrt{-\lambda} |V_\lambda|^2}.$$

Alternatively, if $\rho_{2\mathrm{D}} = \sum_\lambda \rho_\lambda V_\lambda$ then $A_\lambda = \rho_\lambda / (2\epsilon_0 \sqrt{-\lambda})$.

1.3 Beam Slice Potential

After obtaining V_λ with an eigenmode solver of the pipe and the constants A_λ using $\rho_{2\mathrm{D}}$ and the formula in the previous section, the potential anywhere in the pipe is given by

$$V(x, y, z) = \sum_\lambda A_\lambda \mathrm{e}^{-\sqrt{-\lambda}|z|} V_\lambda(x, y).$$

2 Separable Charge Density Case

The result for beam slices generalises to separable charge densities (at rest)

$$\rho = \rho_{2\mathrm{D}}(x, y) f(z),$$

since the problem is linear, translatable in z and $f(z) = \int_{-\infty}^{\infty} f(Z)\delta(z - Z)\,\mathrm{d}Z$,

$$V(x, y, z) = \sum_\lambda A_\lambda V_\lambda(x, y) \int_{-\infty}^{\infty} f(Z) \mathrm{e}^{-\sqrt{-\lambda}|z - Z|}\,\mathrm{d}Z.$$

2.1 Constant Current Beam

Suppose that $f(z)$ is constant; without loss of generality, $f(z) = 1$ with appropriate rescaling of $\rho_{2\mathrm{D}}$. Now

$$\int_{-\infty}^{\infty} f(Z) \mathrm{e}^{-\sqrt{-\lambda}|z - Z|}\,\mathrm{d}Z = \int_{-\infty}^{\infty} \mathrm{e}^{-\sqrt{-\lambda}|z|}\,\mathrm{d}z = 2\int_0^{\infty} \mathrm{e}^{-\sqrt{-\lambda} z}\,\mathrm{d}z = \frac{2}{\sqrt{-\lambda}},$$

so the potential no longer depends on z:

$$V(x, y, z) = \sum_\lambda \frac{2A_\lambda}{\sqrt{-\lambda}} V_\lambda(x, y) = V_{2\mathrm{D}}(x, y) = \frac{1}{\epsilon_0} \sum_\lambda \frac{\rho_{2\mathrm{D}} \cdot V_\lambda}{-\lambda |V_\lambda|^2} V_\lambda(x, y).$$

If $\rho_{\mathrm{2D}} = \sum_\lambda \rho_\lambda V_\lambda$ then by orthogonality, $\rho_{\mathrm{2D}} \cdot V_\lambda = \rho_\lambda |V_\lambda|^2$ and

$$V_{\mathrm{2D}} = \sum_\lambda \frac{\rho_\lambda}{\epsilon_0(-\lambda)} V_\lambda.$$

This can also be derived quickly from the 2D equation $\nabla^2_{x,y} V = -\rho/\epsilon_0$ that assumes $\frac{\mathrm{d}}{\mathrm{d}z} = 0$.

2.2 Longitudinal Space Charge Force

For longitudinal space charge the field component $E_z = -V'$ is of interest and

$$\begin{aligned} V'(x,y,z) &= \sum_\lambda A_\lambda V_\lambda(x,y)\sqrt{-\lambda}\left(-\int_{-\infty}^{z} f(Z)\mathrm{e}^{\sqrt{-\lambda}(Z-z)}\,\mathrm{d}Z + \int_z^\infty f(Z)\mathrm{e}^{\sqrt{-\lambda}(z-Z)}\,\mathrm{d}Z\right) \\ &= \sum_\lambda A_\lambda V_\lambda(x,y)\sqrt{-\lambda}\int_{-\infty}^{\infty} f(Z)\,\mathrm{sgn}(Z-z)\,\mathrm{e}^{-\sqrt{-\lambda}|Z-z|}\,\mathrm{d}Z. \end{aligned}$$

2.3 Derivative of Line Density Approximation

Since $V' = 0$ for $f(z) = 1$, by linearity constant contributions to f can be neglected when calculating E_z. Also, V' is only sensitive to the value of f 'locally', that is in the region a few times $1/\sqrt{-\lambda}$ (the exponential decay length) from z. This means the next Taylor series term $f(Z) = f'(z)(Z-z)$ can be used as an approximation. Note that defining the line density $\rho_{\mathrm{1D}}(z) = \iint \rho(x,y,z)\,\mathrm{d}x\,\mathrm{d}y = f(z)\iint \rho_{\mathrm{2D}}$, we have $f'(z) = \rho'_{\mathrm{1D}}(z)/\iint \rho_{\mathrm{2D}}$. The integral in the last section becomes:

$$\begin{aligned} \int_{-\infty}^{\infty} f(Z)\,\mathrm{sgn}(Z-z)\,\mathrm{e}^{-\sqrt{-\lambda}|Z-z|}\,\mathrm{d}Z &= f'(z)\int_{-\infty}^{\infty}(Z-z)\mathrm{sgn}(Z-z)\,\mathrm{e}^{-\sqrt{-\lambda}|Z-z|}\,\mathrm{d}Z \\ &= f'(z)\int_{-\infty}^{\infty}|Z-z|\mathrm{e}^{-\sqrt{-\lambda}|Z-z|}\,\mathrm{d}Z \\ &= 2f'(z)\int_0^\infty Z\mathrm{e}^{-\sqrt{-\lambda}Z}\,\mathrm{d}Z \\ &= 2f'(z)\left[\left(-\frac{Z}{\sqrt{-\lambda}} - \frac{1}{-\lambda}\right)\mathrm{e}^{-\sqrt{-\lambda}Z}\right]_{Z=0}^{\infty} \\ &= 2f'(z)\frac{1}{-\lambda}. \end{aligned}$$

So the potential is approximately

$$V'(x,y,z) = 2f'(z)\sum_\lambda \frac{A_\lambda}{\sqrt{-\lambda}}V_\lambda(x,y) \quad = \quad f'(z)V_{\mathrm{2D}}(x,y),$$

that is $f'(z)$ times the formula for V in the constant current case. Substituting the formula for A_λ gives

$$V'(x,y,z) = f'(z)\sum_\lambda \frac{\rho_{\mathrm{2D}}\cdot V_\lambda}{\epsilon_0(-\lambda)|V_\lambda|^2}V_\lambda(x,y)$$

and

$$\frac{E_z}{\rho'_{\mathrm{1D}}} = \frac{-V'}{f'(z)\iint\rho_{\mathrm{2D}}} = \frac{1}{\epsilon_0\iint\rho_{\mathrm{2D}}}\sum_\lambda \frac{\rho_{\mathrm{2D}}\cdot V_\lambda}{\lambda|V_\lambda|^2}V_\lambda(x,y) \quad = \quad \frac{-V_{\mathrm{2D}}(x,y)}{\iint\rho_{\mathrm{2D}}}.$$

This still depends on x and y. To further approximate, an averaged value over the whole beam can be obtained by dotting this formula with $\rho_{\mathrm{2D}}/\iint\rho_{\mathrm{2D}}$:

$$\left\langle\frac{E_z}{\rho'_{\mathrm{1D}}}\right\rangle = \frac{1}{\epsilon_0(\iint\rho_{\mathrm{2D}})^2}\sum_\lambda\frac{(\rho_{\mathrm{2D}}\cdot V_\lambda)^2}{\lambda|V_\lambda|^2} \quad = \quad \frac{-\rho_{\mathrm{2D}}\cdot V_{\mathrm{2D}}}{(\iint\rho_{\mathrm{2D}})^2}.$$

2.4 Relationship to the g Factor

A conventional way of writing the longitudinal space charge field (e.g. [1]) is

$$E_z = E_{z,\text{wall}} - \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2}\left(1 + 2\ln\frac{r_{\text{wall}}}{r_{\text{beam}}}\right)\frac{\partial\lambda}{\partial z},$$

where q is the charge on an individual particle, γ is the gamma factor of a moving beam and λ is the particle line density. So far this report has considered perfectly-conducting walls where $E_{z,\text{wall}} = 0$ and stationary 'beams' for which $\gamma = 1$. The charge line density $\rho_{1\text{D}} = q\lambda$. In this case,

$$E_z = -\frac{1}{4\pi\epsilon_0}\left(1 + 2\ln\frac{r_{\text{wall}}}{r_{\text{beam}}}\right)\rho'_{1\text{D}},$$

where the dimensionless term in brackets is called the 'g factor'. This formula assumes a circular pipe and the g factor differs for other shapes. In general, this may be related to the quantity in the previous section via

$$E_z = -\frac{1}{4\pi\epsilon_0}g\rho'_{1\text{D}} \qquad \Rightarrow \qquad g = -4\pi\epsilon_0\left\langle\frac{E_z}{\rho'_{1\text{D}}}\right\rangle \quad = \quad \frac{4\pi}{(\iint\rho_{2\text{D}})^2}\sum_\lambda\frac{(\rho_{2\text{D}}\cdot V_\lambda)^2}{-\lambda|V_\lambda|^2}.$$

In terms of the 2D potential,

$$g = 4\pi\epsilon_0\frac{\rho_{2\text{D}}\cdot V_{2\text{D}}}{(\iint\rho_{2\text{D}})^2} \qquad \text{and} \qquad \langle E_z\rangle = -\rho'_{1\text{D}}\frac{\rho_{2\text{D}}\cdot V_{2\text{D}}}{(\iint\rho_{2\text{D}})^2}.$$

2.5 Example: Circular Pipe

For a circularly symmetric beam in a circular pipe, $\rho_{2\text{D}}\cdot V_\lambda = 0$ for eigenpotentials that vary with θ, leaving only those expressable as $V(r)$. The eigenpotentials must satisfy

$$\nabla^2_{x,y}V(r) = \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = \lambda V \qquad \Rightarrow \qquad r^2\frac{\partial^2 V}{\partial r^2} + r\frac{\partial V}{\partial r} - \lambda r^2 V = 0,$$

where $V'(0) = 0$ by differentiability at the centre and $V(R) = 0$ boundary condition for a pipe of radius R. Changing to a rescaled variable x with $x^2 = -\lambda r^2$ turns this into Bessel's equation

$$x^2\frac{\partial^2 V}{\partial x^2} + x\frac{\partial V}{\partial x} + (x^2 - \alpha^2)V = 0$$

for $\alpha = 0$, with boundary conditions $V'(0) = 0$ and $V(x = \sqrt{-\lambda}R) = 0$. This has the solution $V = J_0(x) = J_0(\sqrt{-\lambda}r)$ but to satisfy the boundary condition, $\sqrt{-\lambda}R = J_{0n}$ for some n, where J_{0n} is the n^{th} root of J_0 (for which there is no analytic expression). Thus the eigenvalues are $\lambda_n = -(J_{0n}/R)^2$ and V_{λ_n} will be abbreviated to $V_n(r) = J_0((r/R)J_{0n})$.

The established result $1 + 2\ln(R/r_{\text{beam}})$ for a circular uniform beam is actually calculated for the field at the centre of the pipe and not averaged over the beam, so define the position-dependent g factor as

$$g(x,y) = -4\pi\epsilon_0\frac{E_z(x,y)}{\rho'_{1\text{D}}} \quad = \quad \frac{4\pi}{\iint\rho_{2\text{D}}}\sum_\lambda\frac{\rho_{2\text{D}}\cdot V_\lambda}{-\lambda|V_\lambda|^2}V_\lambda(x,y) \quad = \quad 4\pi\epsilon_0\frac{V_{2\text{D}}(x,y)}{\iint\rho_{2\text{D}}}.$$

2.5.1 Circular Uniform Beam

Assume a uniform, normalised beam for which $\rho_{2D} = 1/(\pi r_{\text{beam}}^2)$ for $r \le r_{\text{beam}}$ and zero outside. Noting $\iint \rho_{2D} = 1$ and $J_0(0) = 1$, the value of interest is

$$g(0,0) = 4\pi \sum_n \frac{\rho_{2D} \cdot V_n}{-\lambda_n |V_n|^2} V_n(r=0) \quad = \quad 4\pi \sum_n \frac{\rho_{2D} \cdot V_n}{(J_{0n}/R)^2 |V_n|^2}.$$

Consider the dot product

$$\rho_{2D} \cdot V_n = \frac{1}{\pi r_{\text{beam}}^2} \int_{r=0}^{r_{\text{beam}}} J_0((r/R)J_{0n})\, 2\pi r \,\mathrm{d}r \quad = \quad \frac{2}{r_{\text{beam}}^2} \int_{r=0}^{r_{\text{beam}}} J_0((r/R)J_{0n})\, r \,\mathrm{d}r,$$

rescale the integral with $x = (r/R)J_{0n}$, giving $r\,\mathrm{d}r = (R/J_{0n})^2\, x\,\mathrm{d}x$:

$$\rho_{2D} \cdot V_n = \frac{2}{((r_{\text{beam}}/R)J_{0n})^2} \int_{x=0}^{\frac{r_{\text{beam}}}{R} J_{0n}} x J_0(x)\,\mathrm{d}x.$$

Properties of the Bessel functions give that $xJ_0(x)$ is the derivative of $xJ_1(x)$, so

$$\rho_{2D} \cdot V_n = \frac{2}{(r_{\text{beam}}/R)J_{0n}} J_1((r_{\text{beam}}/R)J_{0n}) \quad = \quad \frac{2}{f J_{0n}} J_1(f J_{0n}),$$

where the beam radius fraction $f = r_{\text{beam}}/R$ has been defined (so the textbook result $1 + 2\ln(R/r_{\text{beam}})$ becomes $1 - 2\ln f$). The other integral to calculate is

$$|V_n|^2 = \int_{r=0}^{R} J_0((r/R)J_{0n})^2\, 2\pi r\,\mathrm{d}r \quad = \quad 2\pi R^2 \int_{x=0}^{1} x J_0(xJ_{0n})^2\,\mathrm{d}x,$$

where the substitution $x = r/R$ and $r\,\mathrm{d}r = R^2\, x\,\mathrm{d}x$ has been used. Compare with the orthogonality relation for Bessel functions

$$\int_0^1 x J_\alpha(xJ_{\alpha m}) J_\alpha(xJ_{\alpha n})\,\mathrm{d}x = \frac{\delta_{mn}}{2} J_{\alpha+1}(J_{\alpha m})^2,$$

which for $\alpha = 0$ and $m = n$ becomes

$$\int_0^1 x J_0(xJ_{0n})^2\,\mathrm{d}x = \frac{1}{2} J_1(J_{0n})^2,$$

therefore

$$|V_n|^2 = \pi R^2 J_1(J_{0n})^2.$$

Substituting these into the formula gives

$$g(0,0) = 4\pi \sum_n \frac{\frac{2}{fJ_{0n}} J_1(fJ_{0n})}{(J_{0n}/R)^2 \pi R^2 J_1(J_{0n})^2} \quad = \quad 8 \sum_n \frac{J_1(fJ_{0n})}{f J_{0n}^3 J_1(J_{0n})^2}.$$

The beam-averaged g factor in this case just has an additional factor of $\rho_{2D} \cdot V_n$:

$$g = 16 \sum_n \frac{J_1(fJ_{0n})^2}{f^2 J_{0n}^4 J_1(J_{0n})^2}.$$

Numerical evaluation (30000 terms) shows that to within 10^{-7}, this $g(0,0) = 1 - 2\ln f$ as required and $g = \frac{1}{2} - 2\ln f$.

2.5.2 Circular Waterbag Beam

From results in the last section, for any beam normalised by $\iint \rho_{\text{2D}} = 1$ in a circular pipe,

$$g = 4 \sum_n \frac{(\rho_{\text{2D}} \cdot V_n)^2}{J_{0n}^2 J_1(J_{0n})^2}$$

is the averaged g factor, where $V_n = J_0((r/R)J_{0n})$. For a waterbag beam, $\rho_{\text{2D}} \propto 1 - (r/r_{\text{beam}})^2$ for $r \le r_{\text{beam}}$ and zero outside, so when normalised, $\rho_{\text{2D}} = (2/(\pi r_{\text{beam}}^2))(1 - (r/r_{\text{beam}})^2)$. The term that needs calculating is

$$\begin{aligned}\rho_{\text{2D}} \cdot V_n &= \frac{2}{\pi r_{\text{beam}}^2} \int_{r=0}^{r_{\text{beam}}} (1 - (r/r_{\text{beam}})^2)\, J_0((r/R)J_{0n})\, 2\pi r\, \mathrm{d}r \\ &= \frac{4}{r_{\text{beam}}^2} \int_{r=0}^{r_{\text{beam}}} (1 - (r/r_{\text{beam}})^2)\, J_0((r/R)J_{0n})\, r\, \mathrm{d}r.\end{aligned}$$

Now perform the same rescaling as before with $x = (r/R)J_{0n}$, $r\,\mathrm{d}r = (R/J_{0n})^2\, x\,\mathrm{d}x$ and $r/r_{\text{beam}} = x/(fJ_{0n})$:

$$\begin{aligned}\rho_{\text{2D}} \cdot V_n &= \frac{4R^2}{r_{\text{beam}}^2 J_{0n}^2} \int_{x=0}^{fJ_{0n}} (1 - (x/(fJ_{0n}))^2)\, J_0(x)\, x\,\mathrm{d}x \\ &= \frac{4}{f^2 J_{0n}^2} \int_{x=0}^{fJ_{0n}} xJ_0(x) - \frac{1}{f^2 J_{0n}^2}\, x^3 J_0(x)\,\mathrm{d}x.\end{aligned}$$

Since $J_0' = -J_1$ and $(xJ_1)' = xJ_0$, it can be seen that $(x^3 - 4x)J_1 + 2x^2 J_0$ differentiates to $x^3 J_0$. Therefore

$$\begin{aligned}\rho_{\text{2D}} \cdot V_n &= \frac{4}{f^2 J_{0n}^2} \left[xJ_1(x) - \frac{(x^3 - 4x)J_1(x) + 2x^2 J_0(x)}{f^2 J_{0n}^2} \right]_{x=0}^{fJ_{0n}} \\ &= \frac{4}{f^2 J_{0n}^2} \left(fJ_{0n} J_1(fJ_{0n}) - \frac{((fJ_{0n})^3 - 4fJ_{0n})J_1(fJ_{0n}) + 2(fJ_{0n})^2 J_0(fJ_{0n})}{f^2 J_{0n}^2} \right) \\ &= \frac{4}{f^2 J_{0n}^2} \left(fJ_{0n} J_1(fJ_{0n}) - fJ_{0n} J_1(fJ_{0n}) - 2J_0(fJ_{0n}) + \frac{4J_1(fJ_{0n})}{fJ_{0n}} \right) \\ &= \frac{8}{f^2 J_{0n}^2} \left(\frac{2J_1(fJ_{0n})}{fJ_{0n}} - J_0(fJ_{0n}) \right) \qquad \left(\text{c.f.} \quad \frac{2}{fJ_{0n}} J_1(fJ_{0n}) \quad \text{for uniform beam} \right).\end{aligned}$$

Substituting back into the formula for g gives

$$g = 256 \sum_n \frac{\left(\frac{2J_1(fJ_{0n})}{fJ_{0n}} - J_0(fJ_{0n}) \right)^2}{f^4 J_{0n}^6 J_1(J_{0n})^2}.$$

Numerical computation (30000 terms) shows that to within 10^{-11}, $g = \frac{11}{12} - 2\ln f$.

2.5.3 Potential Method for General Circularly-Symmetric Beam

If the constant current potential V_{2D} is known, the g factor $4\pi\epsilon_0(\rho_{\text{2D}} \cdot V_{\text{2D}})/(\iint \rho_{\text{2D}})^2$ can be found by calculating $\rho_{\text{2D}} \cdot V_{\text{2D}}$ directly. In the case of a circular pipe with perfectly conducting boundary at $r = R$, $V_{\text{2D}} = V(r)$ is determined by

$$\nabla_{x,y}^2 V(r) = \frac{1}{r} V' + V'' = -\frac{\rho}{\epsilon_0}, \qquad V(R) = 0,$$

where prime denotes differentiation by r not z here. The integral to be evaluated is

$$\rho_{2\mathrm{D}} \cdot V_{2\mathrm{D}} = \int_0^R \rho(r) V(r) 2\pi r \,\mathrm{d}r$$

and while there is no general simplification of this, Baartman provides solutions for a wide range of cases using this method in [2].

2.6 Example: Rectangular Pipe

A rectangular beam pipe $[0, X] \times [0, Y]$ has eigenpotentials

$$V_{nm} = \sin(n\pi x/X)\sin(m\pi y/Y) \qquad \text{with} \qquad \lambda_{nm} = -(n\pi/X)^2 - (m\pi/Y)^2$$

for integers $n, m \geq 1$. Calculations will also require the value

$$|V_{nm}|^2 = \int_0^X \int_0^Y V_{nm}^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^X \sin^2(n\pi x/X)\,\mathrm{d}x \int_0^Y \sin^2(m\pi y/Y)\,\mathrm{d}y = \frac{XY}{4}.$$

Substituting these into the g factor formula gives

$$g = \frac{16}{\pi XY(\iint \rho_{2\mathrm{D}})^2} \sum_{n,m\geq 1} \frac{(\rho_{2\mathrm{D}} \cdot V_{nm})^2}{(n/X)^2 + (m/Y)^2}.$$

2.6.1 Uniform Rectangular Beam

Suppose the beam occupies the rectangle $[x_0, x_1] \times [y_0, y_1]$ with $\rho_{2\mathrm{D}} = 1$, so that $\iint \rho_{2\mathrm{D}} = (x_1 - x_0)(y_1 - y_0) = \Delta x \Delta y$. Calculate

$$\begin{aligned}
\rho_{2\mathrm{D}} \cdot V_{nm} &= \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sin(n\pi x/X)\sin(m\pi y/Y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \left[\frac{-X}{n\pi}\cos(n\pi x/X)\right]_{x_0}^{x_1} \left[\frac{-Y}{m\pi}\cos(m\pi y/Y)\right]_{y_0}^{y_1} \\
&= \frac{XY}{nm\pi^2}(\cos(n\pi x_1/X) - \cos(n\pi x_0/X))(\cos(m\pi y_1/Y) - \cos(m\pi y_0/Y)).
\end{aligned}$$

So that

$$g = \frac{16XY}{\pi^5 \Delta x^2 \Delta y^2} \sum_{n,m\geq 1} \frac{(\cos(n\pi x_1/X) - \cos(n\pi x_0/X))^2(\cos(m\pi y_1/Y) - \cos(m\pi y_0/Y))^2}{n^2 m^2((n/X)^2 + (m/Y)^2)}.$$

2.6.2 Elliptical Gaussian Beam

An elliptical Gaussian beam has $\rho_{2\mathrm{D}} = \mathrm{e}^{-(ax^2+2bxy+cy^2)} = \mathrm{e}^{-\mathbf{x}^T\mathbf{M}\mathbf{x}}$ where $\mathbf{M} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. It is well known that $\iint \mathrm{e}^{-\mathbf{x}^T\mathbf{x}} = \iint \mathrm{e}^{-(x^2+y^2)} = \pi$, so composing an arbitrary linear function $\mathbf{x} \to \mathbf{Q}\mathbf{x}$ gives $\iint \mathrm{e}^{-\mathbf{x}^T\mathbf{Q}^T\mathbf{Q}\mathbf{x}} = \pi/\det\mathbf{Q}$, thus if $\mathbf{Q}^T\mathbf{Q} = \mathbf{M}$ then $(\det\mathbf{Q})^2 = \det\mathbf{M}$ and $\iint \mathrm{e}^{-\mathbf{x}^T\mathbf{M}\mathbf{x}} = \pi/\sqrt{\det\mathbf{M}}$, that is $\iint \rho_{2\mathrm{D}} = \pi/\sqrt{ac - b^2}$.

For a Gaussian centred on $(\bar{x}, \bar{y})$ that is not significantly outside the beam pipe, make the approximation to integrate over the whole x–y plane

$$\begin{aligned}\rho_{2\mathrm{D}} \cdot V_{nm} &= \int_{-\bar{x}}^{X-\bar{x}} \int_{-\bar{y}}^{Y-\bar{y}} \mathrm{e}^{-(ax^2+2bxy+cy^2)} \sin(n\pi(\bar{x}+x)/X) \sin(m\pi(\bar{y}+y)/Y)\,\mathrm{d}x\,\mathrm{d}y \\ &\simeq \iint \mathrm{e}^{-(ax^2+2bxy+cy^2)} \sin(n\pi(\bar{x}+x)/X) \sin(m\pi(\bar{y}+y)/Y).\end{aligned}$$

Note that $\sin x = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x})$ so that

$$\begin{aligned}\rho_{2\mathrm{D}} \cdot V_{nm} &\simeq \frac{1}{-4} \sum_{j,k=-1,1} \iint \mathrm{e}^{-(ax^2+2bxy+cy^2)} j\mathrm{e}^{jin\pi(\bar{x}+x)/X} k\mathrm{e}^{kim\pi(\bar{y}+y)/Y} \\ &= -\frac{1}{4} \sum_{j,k=-1,1} jk\mathrm{e}^{\mathrm{i}\pi(jn\bar{x}/X+km\bar{y}/Y)} \iint \mathrm{e}^{-(ax^2+2bxy+cy^2-(\mathrm{i}\pi jn/X)x-(\mathrm{i}\pi km/Y)y)}.\end{aligned}$$

The negative exponent is of the form $\mathbf{x}^T\mathbf{M}\mathbf{x}+\mathbf{b}^T\mathbf{x}$, where $\mathbf{b}^T = -\mathrm{i}\pi[jn/X, km/Y]$. Noting that $\mathbf{M} = \mathbf{M}^T$, a shift of origin expands as $(\mathbf{x}+\mathbf{c})^T\mathbf{M}(\mathbf{x}+\mathbf{c}) = \mathbf{x}^T\mathbf{M}\mathbf{x} + 2\mathbf{c}^T\mathbf{M}\mathbf{x} + \mathbf{c}^T\mathbf{M}\mathbf{c}$. This is equal to $\mathbf{x}^T\mathbf{M}\mathbf{x} + \mathbf{b}^T\mathbf{x} + \mathbf{c}^T\mathbf{M}\mathbf{c}$ if one puts $\mathbf{c} = \frac{1}{2}\mathbf{M}^{-1}\mathbf{b}$. Thus $\mathbf{x}^T\mathbf{M}\mathbf{x} + \mathbf{b}^T\mathbf{x} = (\mathbf{x}+\mathbf{c})^T\mathbf{M}(\mathbf{x}+\mathbf{c}) - \frac{1}{4}\mathbf{b}^T\mathbf{M}^{-1}\mathbf{b}$. The shift won't affect the integral while the constant term in the exponent becomes a constant factor:

$$\begin{aligned}\rho_{2\mathrm{D}} \cdot V_{nm} &\simeq -\frac{1}{4} \sum_{j,k=-1,1} jk\mathrm{e}^{\mathrm{i}\pi(jn\bar{x}/X+km\bar{y}/Y)} \mathrm{e}^{\frac{1}{4}\mathbf{b}^T\mathbf{M}^{-1}\mathbf{b}} \iint \mathrm{e}^{-(ax^2+2bxy+cy^2)} \\ &= -\frac{\pi}{4\sqrt{ac-b^2}} \sum_{j,k=-1,1} jk\mathrm{e}^{\mathrm{i}\pi(jn\bar{x}/X+km\bar{y}/Y)} \mathrm{e}^{\frac{1}{4}\mathbf{b}^T\mathbf{M}^{-1}\mathbf{b}}.\end{aligned}$$

To evaluate the exponent, note that $\mathbf{M}^{-1} = \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$ so

$$\begin{aligned}\frac{1}{4}\mathbf{b}^T\mathbf{M}^{-1}\mathbf{b} &= -\frac{\pi^2}{4}[jn/X, km/Y]\frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}\begin{bmatrix} jn/X \\ km/Y \end{bmatrix} \\ &= -\frac{\pi^2}{4(ac-b^2)}\left(c(jn/X)^2 - 2b(jn/X)(km/Y) + a(km/Y)^2\right) \\ &= -\frac{\pi^2}{4(ac-b^2)}\left(c(n/X)^2 - 2bjk(n/X)(m/Y) + a(m/Y)^2\right),\end{aligned}$$

where the last line has used $j^2 = k^2 = 1$. The summand apart from the $\mathrm{e}^{\mathrm{i}\pi C}$ term now only depends on jk, which shall be renamed to j and collect pairs of terms which differ by both j and k changing sign, which inverts the sign of C, making $\mathrm{e}^{\mathrm{i}\pi C} + \mathrm{e}^{-\mathrm{i}\pi C} = 2\cos\pi C$.

$$\begin{aligned}\rho_{2\mathrm{D}} \cdot V_{nm} &\simeq -\frac{\pi}{2\sqrt{ac-b^2}} \sum_{j=-1,1} j\cos\pi C \exp E \\ &= -\frac{1}{2}\iint \rho_{2\mathrm{D}} \sum_{j=-1,1} j\cos\pi C \exp E,\end{aligned}$$

where $C = n\bar{x}/X + jm\bar{y}/Y$ and $E = -\frac{\pi^2}{4(ac-b^2)}\left(c(n/X)^2 - 2bj(n/X)(m/Y) + a(m/Y)^2\right)$. Aside from the cancellation of $\iint \rho_{2\mathrm{D}}$ no further simplification is possible on substituting into the g factor formula.

3 Moving Beams (Longitudinal Space Charge)

Previous sections have only considered a stationary beam, or alternatively a beam in its co-moving frame (assuming negligible relative velocities between particles). Writing the co-moving quantities with a tilde, the g factor relationship is $\tilde{E}_z = -\frac{1}{4\pi\epsilon_0} g \tilde{\rho}'_{1\mathrm{D}}$. The velocity of the beam is parallel to the z axis, so the parallel component of the electric field $E_z = \tilde{E}_z$ is unchanged under frame transformation and produces no other field components.

The line density is less trivial to transform. Define $Q(z) = \int_{-\infty}^{z} \rho_{1\mathrm{D}}(Z)\,\mathrm{d}Z$ to be the amount of charge at positions $Z \leq z$. Q transforms as a scalar quantity whereas the density $\rho_{1\mathrm{D}} = \frac{\mathrm{d}Q}{\mathrm{d}z}$ is actually a rank 1 covariant tensor and $\rho'_{1\mathrm{D}} = \frac{\mathrm{d}^2 Q}{\mathrm{d}z^2}$ is rank 2 covariant. As the beam is larger in its rest frame by $\tilde{z} = \gamma z$, this means $\rho'_{1\mathrm{D}} = (\frac{\mathrm{d}\tilde{z}}{\mathrm{d}z})^2 \frac{\mathrm{d}^2 Q}{\mathrm{d}\tilde{z}^2} = \gamma^2 \tilde{\rho}'_{1\mathrm{D}}$.

Putting this all together gives the g factor relation for a moving beam as

$$E_z = -\frac{1}{4\pi\epsilon_0}\frac{g}{\gamma^2}\rho'_{1\mathrm{D}}.$$

3.1 Space Charge Energy Gain (or loss) per Turn

If a turn in the machine has length L, then $\Delta E = L\frac{\mathrm{d}E}{\mathrm{d}z} = LF_z = LqE_z$, therefore

$$\Delta E = -\frac{Lq}{4\pi\epsilon_0}\frac{g}{\gamma^2}\rho'_{1\mathrm{D}}.$$

This is equivalent to a voltage of

$$V = \frac{L}{4\pi\epsilon_0}\frac{g}{\gamma^2}\rho'_{1\mathrm{D}}.$$

References

[1] *Particle Accelerator Physics II*, 2nd Edition, H. Wiedemann, section 10.2.4 *Longitudinal Space-Charge Field*, equation (10.55).

[2] *Form Factor g In Longitudinal Space Charge Impedance*, R. Baartman, TRIUMF design note 1992-TRI-DN-K206, available from `http://lin12.triumf.ca/text/design_notes/k206/k206h.pdf`.