Fields of Moving, Non-Accelerating Charges in the Present Time

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1 Motivation

A common approximation in codes that calculate the mutual repulsion of beam particles (space charge) is to assume the 'average' rest frame of the beam using a Lorentz transformation and calculate repulsion based on a potential derived from Coulomb's law. Typically Coulomb's law is too slow for direct evaluation so FEM, FFT or multipole/tree methods are used to speed up the calculation; the potential also needs to be smoothed appropriately in the case of macroparticles. However in all these cases the particles are assumed not to be in relative motion, so no significant $\mathbf{B}$ field is produced in the beam frame.

The correct forces for particles in arbitrary relative motion come from the retarded potential of Maxwell's equations, which although it produces both the $\mathbf{E}$ and $\mathbf{B}$ fields, is difficult to use in computer codes because it requires the locations of particles in the past. The compromise presented in this note is to assume the particles have *small accelerations* rather than small relative velocities, which makes finding the fields a matter of transforming a forever-stationary particle's $\mathbf{E}$ field to the fields of a particle that has always been moving at constant velocity.

2 Transforming the Coulomb Field

A particle of charge q forever at rest at the origin of its rest frame $\mathbf{x}' = \mathbf{0}$ produces the field

$$\mathbf{E}'(\mathbf{x}', t') = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{x}'}{|\mathbf{x}'|^3}, \qquad \mathbf{B}'(\mathbf{x}', t') = \mathbf{0}.$$

If the particle is moving with velocity $\mathbf{v}$, the laboratory frame fields may be recovered using the (inverse) transformation law

$$\mathbf{E} = \gamma(\mathbf{E}' - \mathbf{v} \times \mathbf{B}') - (\gamma - 1)(\mathbf{E}' \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \qquad \mathbf{B} = \gamma\left(\mathbf{B}' + \frac{\mathbf{v} \times \mathbf{E}'}{c^2}\right) - (\gamma - 1)(\mathbf{B}' \cdot \hat{\mathbf{v}})\hat{\mathbf{v}},$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$, $\gamma = (1 - |\mathbf{v}|^2/c^2)^{-1/2}$ and fields are evaluated at the same point, which means $(\mathbf{E}, \mathbf{B})(\mathbf{x}, t)$ and $(\mathbf{E}', \mathbf{B}')(\mathbf{x}', t')$. The laboratory frame is defined so that the fields are being evaluated at position $\mathbf{x}$ at time $t = 0$. The Lorentz transformation gives

$$\mathbf{x}' = \mathbf{x} + (\gamma - 1)(\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \qquad t' = -\gamma\frac{\mathbf{x} \cdot \mathbf{v}}{c^2}.$$

Substituting these into the Coulomb field and defining $r = |\mathbf{x}'| = |\mathbf{x} + (\gamma - 1)(\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}|$ gives

$$\mathbf{E}' = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{x} + (\gamma - 1)(\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}}{r^3}, \qquad \mathbf{B}' = \mathbf{0}$$

and therefore

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^3}(\gamma\mathbf{x}' - (\gamma - 1)(\mathbf{x}' \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}), \qquad \mathbf{B} = \frac{q}{4\pi\epsilon_0 r^3}\gamma\frac{\mathbf{v} \times \mathbf{x}'}{c^2}.$$

Note that $\mathbf{x}' \cdot \hat{\mathbf{v}} = \gamma \mathbf{x} \cdot \hat{\mathbf{v}}$ and $\mathbf{v} \times \mathbf{x}' = \mathbf{v} \times \mathbf{x}$ so that

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^3}(\gamma\mathbf{x} + \gamma(\gamma-1)(\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} - (\gamma-1)\gamma(\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}) = \frac{q}{4\pi\epsilon_0 r^3}\gamma\mathbf{x}$$

and

$$\mathbf{B} = \frac{q}{4\pi\epsilon_0 r^3}\gamma\frac{\mathbf{v}\times\mathbf{x}}{c^2}.$$

3 Evaluation on a Computer

A square root can be avoided by never calculating $\hat{\mathbf{v}}$ explicitly. Instead, calculate $|\mathbf{v}|^2$ and then γ (which has an unavoidable square root) and use

$$r^2 = \left|\mathbf{x} + \frac{\gamma-1}{|\mathbf{v}|^2}(\mathbf{x}\cdot\mathbf{v})\mathbf{v}\right|^2.$$

Calculating r^3 from r^2 involves another unavoidable square root.

4 Summation

For a multi-particle beam, the fields in the laboratory frame from individual particles may be summed:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i\gamma_i(\mathbf{x}-\mathbf{x}_i)}{r_i^3}, \qquad \mathbf{B}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0 c^2}\sum_i \frac{q_i\gamma_i\mathbf{v}_i\times(\mathbf{x}-\mathbf{x}_i)}{r_i^3}.$$

Here, particle i has charge q_i, position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$; $\gamma_i = (1-|\mathbf{v}_i|^2/c^2)^{-1/2}$ and

$$r_i = \left|(\mathbf{x}-\mathbf{x}_i) + \frac{\gamma_i-1}{|\mathbf{v}_i|^2}((\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{v}_i)\mathbf{v}_i\right|$$

is the distance to particle i measured in its rest frame.