

# Off-Axis Magnetic Fields Extrapolated from On-Axis Multipoles

Stephen Brooks

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## 1 Potential Term

In cylindrical polar coordinates  $(r, \theta, z)$ , consider the magnetic scalar potential

$$\phi = \sin(n\theta + \psi)r^k f(z)$$

for some integers  $n, k$ , angle  $\psi$  and function  $f$ . The associated magnetic field is

$$\mathbf{B} = \nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\mathbf{e}_z = \begin{bmatrix} \sin(n\theta + \psi)kr^{k-1}f(z) \\ n\cos(n\theta + \psi)r^{k-1}f(z) \\ \sin(n\theta + \psi)r^k f'(z) \end{bmatrix}_{r\theta z},$$

which automatically satisfies  $\nabla \times \mathbf{B} = \nabla \times \nabla\phi = \mathbf{0}$ . The only remaining condition on  $\phi$  is

$$\begin{aligned} 0 = \nabla \cdot \mathbf{B} &= \nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \\ &= \sin(n\theta + \psi) \left( k(k-1)r^{k-2}f(z) + kr^{k-2}f(z) - n^2r^{k-2}f(z) + r^k f''(z) \right) \\ &= \sin(n\theta + \psi)r^{k-2} \left( (k^2 - n^2)f(z) + r^2 f''(z) \right). \end{aligned}$$

### 1.1 Long Multipole

If there is no  $z$  behaviour ( $f = 1$ , say), then for this to hold for all  $r, \theta$  requires  $k^2 = n^2$ . Since  $k \leq 0$  cases have a singularity at the origin, put  $k = n \geq 1$ . This gives the field for an infinitely long multipole:

$$\phi = \sin(n\theta + \psi)r^n, \quad \mathbf{B} = \begin{bmatrix} n\sin(n\theta + \psi)r^{n-1} \\ n\cos(n\theta + \psi)r^{n-1} \\ 0 \end{bmatrix}_{r\theta z} = nr^{n-1} \begin{bmatrix} \sin((n-1)\theta + \psi) \\ \cos((n-1)\theta + \psi) \\ 0 \end{bmatrix}.$$

$n = 1$  corresponds to a dipole,  $n = 2$  to a quadrupole, etc. Also  $\psi = 0$  gives these in their normal orientation and  $\psi = \frac{\pi}{2}$  in their skew orientation.

### 1.2 Normalisation

Although the rest of the paper will evaluate fields for the potential containing  $r^n$  above, this produces fields with magnitude  $|\mathbf{B}| = nr^{n-1}$ . If multipole strengths are defined as the values of polynomial coefficients of the field function, then  $|\mathbf{B}| = k_n r^n$  may be obtained from the potential  $\phi = \sin(n\theta + \psi)\frac{k_n}{n+1}r^{n+1}$ . If strengths are defined as repeated derivatives of the field function, then  $|d^n \mathbf{B}/dx^n| = d_n$  (i.e.  $|\mathbf{B}| = \frac{d_n}{n!}r^n$ ) may be obtained from  $\phi = \sin(n\theta + \psi)\frac{d_n}{(n+1)!}r^{n+1}$ .

## 2 Series Solution

If  $f''$  is nonzero, the  $\nabla \cdot \nabla \phi = 0$  equation cannot be satisfied by a single term for all  $r$ . However, consider a sum of such terms with the same  $n$  but different  $k$ :

$$\phi = \sin(n\theta + \psi) \sum_{k=n}^{\infty} r^k f_k(z).$$

Note the lowest term with  $k = n$ , which dominates the sum near the  $r = 0$  axis, corresponds to a long  $2n$ -pole modulated by  $f_n(z)$ . The potential must satisfy

$$0 = \nabla \cdot \nabla \phi = \sin(n\theta + \psi) \sum_{k=n}^{\infty} r^{k-2} \left( (k^2 - n^2) f_k(z) + r^2 f_k''(z) \right)$$

for all  $\theta$ , so the RHS sum must be zero (for all  $r, z$ ). Equating coefficients of  $r^k$  gives

$$((k+2)^2 - n^2) f_{k+2}(z) + f_k''(z) = 0$$

for  $k \geq n$  and the remaining coefficients of  $r^{n-2}$  and  $r^{n-1}$  are 0 and  $((n+1)^2 - n^2) f_{n+1}(z)$  respectively, so  $f_{n+1}(z) = 0$ . The above relation gives  $f_{k+2}$  as a scaled second derivative of  $f_k$ , so  $f_{n+2j+1}(z) = 0$  for all  $j \geq 0$  and

$$f_{n+2j}(z) = \left( \prod_{i=1}^j \frac{-1}{(n+2i)^2 - n^2} \right) f_n^{(2j)}(z).$$

The coefficients will be written as

$$C_{nj} = \prod_{i=1}^j \frac{-1}{(n+2i)^2 - n^2} = \prod_{i=1}^j \frac{-1}{4(n+i)i} = \frac{(-\frac{1}{4})^j n!}{(n+j)! j!}.$$

Setting  $f_n = f$ , the full potential satisfying Maxwell's equations in free space is:

$$\phi = \sin(n\theta + \psi) \sum_{j=0}^{\infty} r^{n+2j} f_{n+2j}(z) = \sin(n\theta + \psi) \sum_{j=0}^{\infty} C_{nj} r^{n+2j} f^{(2j)}(z).$$

### 2.1 Magnetic Field

Using the formula for the gradient of a single term  $(\sin(n\theta + \psi) r^k f(z))$  given at the start of this note, the magnetic field associated with the series solution is

$$\begin{aligned} \mathbf{B} = \nabla \phi &= \sum_{j=0}^{\infty} C_{nj} \begin{bmatrix} \sin(n\theta + \psi) (n+2j) r^{n+2j-1} f^{(2j)}(z) \\ n \cos(n\theta + \psi) r^{n+2j-1} f^{(2j)}(z) \\ \sin(n\theta + \psi) r^{n+2j} f^{(2j+1)}(z) \end{bmatrix}_{r\theta z} \\ &= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} (n+2j) \sin(n\theta + \psi) f^{(2j)}(z) \\ n \cos(n\theta + \psi) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix}_{r\theta z} \\ &= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \left( \begin{bmatrix} n \sin(n\theta + \psi) f^{(2j)}(z) \\ n \cos(n\theta + \psi) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix}_{r\theta z} + \begin{bmatrix} 2j \sin(n\theta + \psi) f^{(2j)}(z) \\ 0 \\ 0 \end{bmatrix}_{r\theta z} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \left( \begin{bmatrix} n \sin((n-1)\theta + \psi) f^{(2j)}(z) \\ n \cos((n-1)\theta + \psi) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix} + \begin{bmatrix} 2j \sin(n\theta + \psi) f^{(2j)}(z) \cos \theta \\ 2j \sin(n\theta + \psi) f^{(2j)}(z) \sin \theta \\ 0 \end{bmatrix} \right) \\
&= \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} (n \sin((n-1)\theta + \psi) + 2j \sin(n\theta + \psi) \cos \theta) f^{(2j)}(z) \\ (n \cos((n-1)\theta + \psi) + 2j \sin(n\theta + \psi) \sin \theta) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix}.
\end{aligned}$$

## 2.2 Computation Without Trigonometric Formulae

The product formulae for sin and cos give

$$\begin{aligned}
\sin(n\theta + \psi) \cos \theta &= \frac{1}{2} \sin((n-1)\theta + \psi) + \frac{1}{2} \sin((n+1)\theta + \psi) \\
\sin(n\theta + \psi) \sin \theta &= \frac{1}{2} \cos((n-1)\theta + \psi) - \frac{1}{2} \cos((n+1)\theta + \psi),
\end{aligned}$$

which enables the Cartesian field formula to be written as

$$\mathbf{B} = \sum_{j=0}^{\infty} C_{nj} r^{n+2j-1} \begin{bmatrix} ((n+j) \sin((n-1)\theta + \psi) + j \sin((n+1)\theta + \psi)) f^{(2j)}(z) \\ ((n+j) \cos((n-1)\theta + \psi) - j \cos((n+1)\theta + \psi)) f^{(2j)}(z) \\ r \sin(n\theta + \psi) f^{(2j+1)}(z) \end{bmatrix}.$$

Note that every instance of cos or  $\sin(n\theta + \psi)$  is multiplied by a large power of  $r$  usually including  $r^n$ . This can be used to convert fully to Cartesian coordinates using the complex formulae

$$\begin{aligned}
r e^{i\theta} &= r(\cos \theta + i \sin \theta) = x + iy \\
\Rightarrow (r e^{i\theta})^n &= r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta) = (x + iy)^n,
\end{aligned}$$

whence setting  $c_n = \operatorname{Re}(x + iy)^n$  and  $s_n = \operatorname{Im}(x + iy)^n$  gives

$$r^n \cos n\theta = c_n \quad \text{and} \quad r^n \sin n\theta = s_n.$$

These values can be calculated without trigonometry from the recurrence relation

$$c_0 = 1; \quad s_0 = 0; \quad c_{n+1} = x c_n - y s_n; \quad s_{n+1} = y c_n + x s_n,$$

which comes from the definition of complex multiplication  $(x + iy)^{n+1} = (x + iy)(x + iy)^n$  in terms of components. The formula requires a slightly more general form

$$\hat{c}_n = r^n \cos(n\theta + \psi) = \operatorname{Re} r^n e^{in\theta} e^{i\psi}, \quad \hat{s}_n = r^n \sin(n\theta + \psi) = \operatorname{Im} r^n e^{in\theta} e^{i\psi}.$$

These satisfy a similar recurrence because the repeated multiplication by  $r e^{i\theta} = x + iy$  starts with  $e^{i\psi}$  rather than 1:

$$\hat{c}_0 = \cos \psi; \quad \hat{s}_0 = \sin \psi; \quad \hat{c}_{n+1} = x \hat{c}_n - y \hat{s}_n; \quad \hat{s}_{n+1} = y \hat{c}_n + x \hat{s}_n.$$

The evaluations of  $\cos \psi$  and  $\sin \psi$  are not a problem because they can be precalculated for each multipole, unlike  $\hat{c}_n$  and  $\hat{s}_n$  that depend on  $x, y$ . The field formula can now be rewritten as

$$\begin{aligned}
\mathbf{B} &= \sum_{j=0}^{\infty} C_{nj} r^{2j} \begin{bmatrix} ((n+j) \hat{s}_{n-1} + j \frac{1}{r^2} \hat{s}_{n+1}) f^{(2j)}(z) \\ ((n+j) \hat{c}_{n-1} - j \frac{1}{r^2} \hat{c}_{n+1}) f^{(2j)}(z) \\ \hat{s}_n f^{(2j+1)}(z) \end{bmatrix} \\
&= \sum_{j=0}^{\infty} C_{nj} \begin{bmatrix} ((n+j)(r^2)^j \hat{s}_{n-1} + j(r^2)^{j-1} \hat{s}_{n+1}) f^{(2j)}(z) \\ ((n+j)(r^2)^j \hat{c}_{n-1} - j(r^2)^{j-1} \hat{c}_{n+1}) f^{(2j)}(z) \\ (r^2)^j \hat{s}_n f^{(2j+1)}(z) \end{bmatrix},
\end{aligned}$$

where powers of  $r^2 = x^2 + y^2$  have been emphasised as they can be calculated without a square root. Note that negative powers never arise from  $(r^2)^{j-1}$  when  $j = 0$  because it is multiplied by  $j$ , so disappears.

## 2.3 Computation Using a 2D Field Map

The only parts of the above formula that depend on  $\theta$  are the various  $\hat{c}_n$  and  $\hat{s}_n$ , which are relatively quick (and unavoidable) to calculate for each evaluation point. Calculating the repeated derivatives of  $f$  and waiting for the infinite sum to converge are, on the other hand, quite slow but may be precalculated as three functions of just  $r$  and  $z$  in the following way:

$$\mathbf{B} = \begin{bmatrix} \hat{s}_{n-1}F(r, z) + \hat{s}_{n+1}G(r, z) \\ \hat{c}_{n-1}F(r, z) - \hat{c}_{n+1}G(r, z) \\ \hat{s}_n H(r, z) \end{bmatrix},$$

where

$$F(r, z) = \sum_{j=0}^{\infty} C_{nj}(n+j)(r^2)^j f^{(2j)}(z); \quad G(r, z) = \sum_{j=1}^{\infty} C_{nj}j(r^2)^{j-1} f^{(2j)}(z);$$

$$H(r, z) = \sum_{j=0}^{\infty} C_{nj}(r^2)^j f^{(2j+1)}(z).$$

### 2.3.1 Universal Field Map

Considering just one end of the magnet, if the fringe field is related to a universal fringe field function via  $f(z) = \hat{f}(z/l)$ , then  $f^{(n)}(z) = l^{-n}\hat{f}^{(n)}(z/l)$  and

$$F(r, z) = \sum_{j=0}^{\infty} C_{nj}(n+j) \left(\frac{r}{l}\right)^{2j} \hat{f}^{(2j)}\left(\frac{z}{l}\right) = \hat{F}_n\left(\frac{r}{l}, \frac{z}{l}\right);$$

$$G(r, z) = \frac{1}{r^2} \sum_{j=1}^{\infty} C_{nj}j \left(\frac{r}{l}\right)^{2j} \hat{f}^{(2j)}\left(\frac{z}{l}\right) = \frac{1}{r^2} \hat{G}_n\left(\frac{r}{l}, \frac{z}{l}\right);$$

$$H(r, z) = \frac{1}{r} \sum_{j=0}^{\infty} C_{nj} \left(\frac{r}{l}\right)^{2j+1} \hat{f}^{(2j+1)}\left(\frac{z}{l}\right) = \frac{1}{r} \hat{H}_n\left(\frac{r}{l}, \frac{z}{l}\right),$$

where the new functions  $\hat{F}_n$ ,  $\hat{G}_n$ ,  $\hat{H}_n$  depend only on  $n$  and the form of  $\hat{f}$ , not  $l$ . In terms of these new functions,

$$\mathbf{B} = \begin{bmatrix} \hat{s}_{n-1}\hat{F}_n(r/l, z/l) + \hat{s}_{n+1}\frac{1}{r^2}\hat{G}_n(r/l, z/l) \\ \hat{c}_{n-1}\hat{F}_n(r/l, z/l) - \hat{c}_{n+1}\frac{1}{r^2}\hat{G}_n(r/l, z/l) \\ \hat{s}_n\frac{1}{r}\hat{H}_n(r/l, z/l) \end{bmatrix}.$$

Unfortunately use of the universal function  $\hat{H}_n$  incurs a square root to calculate  $\frac{1}{r}$ , which the non-universal forms do not have, assuming they are stored as functions of  $r^2$  and  $z$ .

### 2.3.2 Field Map for Magnet Ends Only

In the interior of a long magnet, the field tends towards the expression given earlier:

$$\mathbf{B} = nr^{n-1} \begin{bmatrix} \sin((n-1)\theta + \psi) \\ \cos((n-1)\theta + \psi) \\ 0 \end{bmatrix} = \begin{bmatrix} n\hat{s}_{n-1} \\ n\hat{c}_{n-1} \\ 0 \end{bmatrix}.$$

This corresponds to  $F(r, z) = n$  and  $G(r, z) = H(r, z) = 0$ . The lowest-order terms up to  $r^2$  in these sums are:

$$\begin{aligned}
F(r, z) &= C_{n0}nf(z) + C_{n1}(n+1)r^2f''(z) + \dots = nf(z) - \frac{1}{4}r^2f''(z) + \dots; \\
G(r, z) &= C_{n1}f''(z) + C_{n2}2r^2f''''(z) + \dots = \frac{-f''(z)}{4(n+1)} + \frac{r^2f''''(z)}{32(n+1)(n+2)} + \dots; \\
H(r, z) &= C_{n0}f'(z) + C_{n1}r^2f'''(z) + \dots = f'(z) - \frac{r^2f'''(z)}{4(n+1)} + \dots,
\end{aligned}$$

so when  $f(z) = 1$  and  $f'(z) = f''(z) = 0$  to some precision, the field calculation may be replaced by the simple multipole expression. Similarly, when  $f = f' = f'' = 0$  to a good approximation, the field may be given as zero. This means  $F, G, H$  only need to be calculated for the transition region where  $0 < f(z) < 1$  for each magnet (or end of magnet).

### 2.3.3 Magnets with Symmetrical Ends

It is common to use the same fringe field for both ends of a magnet of length  $L$ , so that  $f(z) = g(z) + g(L - z)$ , where  $g$  is a sigmoid function going from  $-\frac{1}{2}$  to  $\frac{1}{2}$ , representing one end of the magnet (e.g.  $g(z) = \frac{1}{2} \tanh(z/l)$ ). If  $F, G, H$  are now calculated using  $g$  in place of  $f$ , the magnetic field is given by

$$\mathbf{B} = \begin{bmatrix} \hat{s}_{n-1}(F(r, z) + F(r, L - z)) + \hat{s}_{n+1}(G(r, z) + G(r, L - z)) \\ \hat{c}_{n-1}(F(r, z) + F(r, L - z)) - \hat{c}_{n+1}(G(r, z) + G(r, L - z)) \\ \hat{s}_n(H(r, z) - H(r, L - z)) \end{bmatrix},$$

noting the sign change for  $H(r, L - z)$  because it contains odd derivatives of  $g$ .

For the regions where  $g(z)$  is almost constant at  $-\frac{1}{2}$  or  $\frac{1}{2}$ ,  $F(r, z) = -\frac{1}{2}n$  or  $\frac{1}{2}n$  may be used respectively and  $G(r, z) = H(r, z) = 0$ . If both  $g(z)$  and  $g(L - z)$  are in a constant region and they have opposite sign (i.e. far outside the magnet), the  $\mathbf{B}$  field will be zero and the calculation should be aborted before calculating  $\hat{s}_{n-1}$  etc.