Exact Tracking in s in a Magnetic Field

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1 Previous Result

In a cartesian coordinate system, a particle of charge q and momentum p in a magnetic field $\mathbf{B}$ will follow a curved path according to

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}z} = \frac{q}{p}\frac{1}{u_z}\mathbf{u} \times \mathbf{B},$$

where $\mathbf{u} = \mathbf{v}/v = \mathbf{p}/p$ is the unit direction vector of the particle's motion [1]. This formula is an exact expression of the Lorentz force law (without electric field) provided that $u_z > 0$.

This may be manipulated [1] to give the exact evolution of $x' = u_x/u_z$ and $y' = u_y/u_z$ with respect to z. At no point has the paraxial approximation $|x'|, |y'| \ll 1$ been used.

2 Curved Coordinate System

Curvature κ in the z–x plane is defined such that positively charged particles experience positive curvature when $B_y > 0$. By the Lorentz force law, if motion is in the $+z$ direction and field is in the $+y$ direction, force on a positive particle will be in the $-x$ direction. This means the $+x$ direction is aligned with the $+r$ radial direction for $\kappa > 0$ (in fact, $r = x + \frac{1}{\kappa}$) when the particle is travelling in the $+z$ direction. Viewed from 'above' ($+y$), positive curvature corresponds to clockwise motion with $\mathrm{d}\theta/\mathrm{d}s = -\kappa$.

In this treatment, $\kappa(s)$ will be a property of the curved reference coordinate system for the entire accelerator and not of any individual particle. The variable s will agree with arc length on the reference curve of the accelerator and planes of constant s will be perpendicular to the reference curve (up to the axis of curvature). The curvilinear coordinate $\tilde{x}$ replaces x, so that the reference curve satisfies $\tilde{x} = y = 0$, with no change needed to y as there is no vertical curvature.

3 Making s the Independent Variable

Without loss of generality, analysis from this point on will assume the curvilinear and cartesian axes are momentarily aligned: $\tilde{x}$ with x and s with z. Considering the forward motion in z given by an increment in s gives $\mathrm{d}z/\mathrm{d}s = 1 + \kappa\tilde{x}$, which equals zero at the centre of curvature $\tilde{x} = -\frac{1}{\kappa}$ as expected.

The evolution of $\mathbf{u}$ can now be restated in terms of s, remembering that the elements of $\mathbf{u}$ are still relative to the fixed cartesian axes:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}s} = (1 + \kappa\tilde{x})\frac{q}{p}\frac{1}{u_z}\mathbf{u} \times \mathbf{B}.$$

4 Rotation of Basis

The vector $\mathbf{w}$ is defined to contain the same direction as $\mathbf{u}$ but expressed in the local $\tilde{x}, y, s$ basis rather than the x, y, z one. At the momentary point of axis-alignment, $\mathbf{w} = \mathbf{u}$ but the derivatives differ by a rotation of rate $-\kappa$:

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} + \begin{bmatrix} \kappa u_z \\ 0 \\ -\kappa u_x \end{bmatrix} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} + \mathbf{u} \times (-\kappa \mathbf{e}_y).$$

5 Conclusion (Vector Form)

Combining the previous two results gives

$$\begin{aligned} \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}s} &= (1 + \kappa\tilde{x})\frac{q}{p}\frac{1}{u_z}\mathbf{u} \times \mathbf{B} + \mathbf{u} \times (-\kappa \mathbf{e}_y) \\ &= \mathbf{u} \times \left((1 + \kappa\tilde{x})\frac{q}{p}\frac{1}{u_z}\mathbf{B} - \kappa \mathbf{e}_y \right) \\ &= \mathbf{w} \times \left((1 + \kappa\tilde{x})\frac{q}{p}\frac{1}{w_s}\mathbf{B} - \kappa \mathbf{e}_y \right), \end{aligned}$$

where the final line uses the momentary axis alignment. As a cartesian basis can be chosen at each point, the analysis holds for all s. Setting $\kappa = 0$ in this formula recovers the result for cartesian coordinates.

5.1 Cyclotron Radius Check

The form of the above equation means that $\mathbf{w}$ is constant if the term in brackets is zero. If a particle is travelling directly along the reference trajectory, $\tilde{x} = 0$ and $w_s = 1$, so $\mathbf{w}$ will be constant (that is, the particle will remain on the reference trajectory) if

$$\frac{q}{p}\mathbf{B} - \kappa \mathbf{e}_y = 0 \qquad \Rightarrow \qquad \mathbf{B} = \frac{p}{q}\kappa \mathbf{e}_y = \frac{p}{qR}\mathbf{e}_y,$$

where $R = 1/\kappa$ is the radius of curvature. The relation $B = p/qR$ is the well-known formula for the cyclotron radius.

6 Geometrical Variables x' and y'

In the curvilinear coordinate system there are analogous definitions of $x' = w_{\tilde{x}}/w_s$ and $y' = w_y/w_s$. Following previous work [1], their derivatives are given by

$$\frac{\mathrm{d}x'}{\mathrm{d}s} = \frac{1}{w_s}\left(\frac{\mathrm{d}w_{\tilde{x}}}{\mathrm{d}s} - x'\frac{\mathrm{d}w_s}{\mathrm{d}s}\right) \qquad \frac{\mathrm{d}y'}{\mathrm{d}s} = \frac{1}{w_s}\left(\frac{\mathrm{d}w_y}{\mathrm{d}s} - y'\frac{\mathrm{d}w_s}{\mathrm{d}s}\right).$$

Next, $\mathrm{d}\mathbf{w}/\mathrm{d}s$ needs to be expressed in terms of x' and y':

$$\frac{1}{w_s}\mathbf{w}\times\mathbf{B} = \begin{bmatrix} y'B_s - B_y \\ B_{\tilde{x}} - x'B_s \\ x'B_y - y'B_{\tilde{x}} \end{bmatrix} \qquad \Rightarrow \qquad \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}s} = (1+\kappa\tilde{x})\frac{q}{p}\begin{bmatrix} y'B_s - B_y \\ B_{\tilde{x}} - x'B_s \\ x'B_y - y'B_{\tilde{x}} \end{bmatrix} - \kappa\begin{bmatrix} -w_s \\ 0 \\ w_{\tilde{x}} \end{bmatrix}.$$

Since $\frac{1}{w_s} = \sqrt{1+x'^2+y'^2}$,

$$\frac{1}{w_s}\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}s} = \sqrt{1+x'^2+y'^2}\,(1+\kappa\tilde{x})\frac{q}{p}\begin{bmatrix} y'B_s - B_y \\ B_{\tilde{x}} - x'B_s \\ x'B_y - y'B_{\tilde{x}} \end{bmatrix} - \kappa\begin{bmatrix} -1 \\ 0 \\ x' \end{bmatrix}.$$

Finally, combinations of the rows of this vector give

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} x' \\ y' \end{bmatrix} = \sqrt{1+x'^2+y'^2}\,(1+\kappa\tilde{x})\frac{q}{p}\begin{bmatrix} y'B_s - B_y - x'^2B_y + x'y'B_{\tilde{x}} \\ B_{\tilde{x}} - x'B_s - x'y'B_y + y'^2B_{\tilde{x}} \end{bmatrix} + \kappa\begin{bmatrix} 1+x'^2 \\ x'y' \end{bmatrix}.$$

The differences with the cartesian formula, apart from relabelling of axes, are the second term and the factor of $1+\kappa\tilde{x}$ in the first term; the cartesian formula is recovered by setting $\kappa = 0$.

6.1 New Derivative of x and y

The variables x' and y' are still the tangents of angles relative to the forward direction, in other words they are *not* equal to $\mathrm{d}\tilde{x}/\mathrm{d}s$ and $\mathrm{d}y/\mathrm{d}s$ but rather $\mathrm{d}\tilde{x}/\mathrm{d}z$ and $\mathrm{d}y/\mathrm{d}z$, recalling that the z axis is momentarily aligned with s but represents a distance. Thus the equations for updating $\tilde{x}$ and y are now

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} \tilde{x} \\ y \end{bmatrix} = \frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \tilde{x} \\ y \end{bmatrix} = (1+\kappa\tilde{x})\begin{bmatrix} x' \\ y' \end{bmatrix}.$$

References

[1] *Exact Tracking in z in a Magnetic Field*, S.J. Brooks, available from `http://stephenbrooks.org/ral/report/2012-2/magnetztracking.pdf` (2012).