Isochronous, Scaling VFFAGs Dominated by Strong Focussing

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1 Introduction

/**/Motivation, VFFAGs, adjustment (cite '62 paper and VEMMA report), relativistic cyclotron-like operation

2 Pure Strong Focussing Case

Assume particles of momentum p have a closed orbit of mean radius $r(p)$ and height $y(p)$. In an isochronous machine the revolution period T is fixed and

$$T = \frac{2\pi r}{\beta c} \quad \Rightarrow \quad r(p) = \frac{\beta(p)cT}{2\pi} = \beta(p)R, \qquad \text{where } R = \frac{cT}{2\pi}.$$

To maintain a mean radius r for particles of momentum p (and charge q and mass m) requires a mean bending field

$$B = \frac{p}{qr} = \frac{m\beta\gamma c}{q\beta R} = \frac{mc}{qR}\,\gamma \quad \Rightarrow \quad B(p) = \gamma(p)B_0, \qquad \text{where } B_0 = \frac{2\pi m}{qT}.$$

If the focussing structure also 'scales', the integrated field gradient when normalised to momentum ($B'l/p$) should be constant per cell. Scaling ensures similarity between optical beta functions at all energies and is a sufficient condition for fixed tunes (but not a necessary condition in lattices with three or more lenses [1]).

The machine is assumed to consist of a large number of cells each occupying angle Θ around the y axis. B' is defined as $\mathrm{d}B_y/ds$ where s is the arc length of the orbit excursion in (r, y). As all bending is about the y axis, there is only a B_y component to the mean bending field.

The assumption of a *large* number of cells ensures that the ring tune is dominated by strong (alternating gradient) focussing and not the single unit of horizontal tune from the coordinate change of rotating around the ring (weak focussing). The analysis is more complicated when weak focussing is included, particularly since the gradient B' is in general at some angle between normal and skew, but the weak focussing remains in the horizontal plane, so there is coupling between transverse planes.

The cell length is given by $l = r\Theta$, so the normalised integrated gradient is

$$\frac{B'l}{p} = \frac{B'r\Theta}{p} = \frac{B'\beta R\Theta}{m\beta\gamma c} = \frac{R\Theta}{mc}\,\frac{B'}{\gamma}.$$

For this to be constant, $B'(p) = \gamma(p) B'_0$ for some gradient B'_0, but it is also true that

$$B' = \frac{\mathrm{d}B}{\mathrm{d}s} = \frac{\mathrm{d}\gamma}{\mathrm{d}s} B_0 = \gamma B'_0 \quad \Rightarrow \quad \frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{B'_0}{B_0}\,\gamma \quad \Rightarrow \quad \gamma = \mathrm{e}^{(B'_0/B_0)s},$$

for an appropriate choice of s origin. This gives the spatial field and gradient profiles

$$B(s) = B_0\,\mathrm{e}^{(B'_0/B_0)s}, \qquad B'(s) = B'_0\,\mathrm{e}^{(B'_0/B_0)s}.$$

Previous VFFAG papers [2, 3] have defined the exponential rate of field increase $k = B'_0/B_0$, which has units of inverse length. Here it will be informative to consider the 'scaling length' in which the field increases by a factor of e instead: $S = 1/k = B_0/B'_0$.

Now that B, B', γ (hence p) and s are all related, all that remains is to find the shape of the orbit excursion in (r, y) space. The link with that space is the radius equation

$$r = \beta R = R\sqrt{1-\gamma^{-2}} = R\sqrt{1-\mathrm{e}^{-2(B'_0/B_0)s}}.$$

Differentiating gives

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}s} &= \frac{R}{2\sqrt{1-\gamma^{-2}}}\; 2(B'_0/B_0)\,\mathrm{e}^{-2(B'_0/B_0)s} \\
&= \frac{R}{2\beta}\; 2(B'_0/B_0)\gamma^{-2} \\
&= \frac{R}{\beta\gamma^2}\,\frac{B'_0}{B_0} \quad = \quad \frac{1}{\beta\gamma^2}\,\frac{R}{S}.
\end{aligned}$$

However, this value tends to infinity at low energy, which is not possible since $\mathrm{d}r/\mathrm{d}s = \cos\theta \leq 1$, where θ is the angle of the orbit excursion with 0 being horizontal and 90° vertical.

2.1 Fundamental Lower Bound on Energy

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{1}{\beta\gamma^2}\,\frac{R}{S} \leq 1 \quad \Rightarrow \quad \beta\gamma^2 \geq \frac{R}{S}.$$

/**/Discuss, table 10,20,50,100,150,200,300,400,500,600,800,1000 MeV, pipe-like machines inefficient, could use in conjuction with cyclotron, no problem with overfocussing as per horizontal isochronous FFAG

2.2 'Obvious' Restriction on Tune Change

/**/Gluing machines together

2.3 Visualising the Orbit Excursion

3 Smooth Focussing Analysis including Weak Focussing

4 Spiral and Other Machine Types

References

[1] *Three-Lens Lattices for Extending the Energy Range of Non-scaling FFAGs*, S.J. Brooks, Proc. IPAC 2011.

[2] *Vertical Orbit Excursion FFAGs*, S.J. Brooks, Proc. HB2010.

[3] *Acceleration in Vertical Orbit Excursion FFAGs with Edge Focussing*, S.J. Brooks, Proc. HB2012.