Analytic Formulae for Fields in Windowframe Magnets

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1 Problem

We will consider an idealised magnetostatics problem where the iron of the windowframe has $\mu_r = \infty$ so produces a perfect infinite family of image currents (or equivalently the transverse $\mathbf{B}$ field component is zero at the surface of the iron). Within the interior of the windowframe, we further assume that $\mu_r = 1$ i.e. $\mu = \mu_0$ and there is no magnetisation, $\mathbf{M} = \mathbf{0}$. Under these assumptions, Maxwell's equations for the $\mathbf{B}$ field become:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}, \end{aligned}$$

where $\mathbf{J}$ is the current density vector.

2 2D Version

Expanding these equations into their components gives:

$$\begin{aligned} \partial_x B_x + \partial_y B_y + \partial_z B_z &= 0 \\ \partial_y B_z - \partial_z B_y &= \mu_0 J_x \\ \partial_z B_x - \partial_x B_z &= \mu_0 J_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z. \end{aligned}$$

In a 2D problem, there is no z variation, so $\partial_z = 0$, simplifying the equations to:

$$\begin{aligned} \partial_x B_x + \partial_y B_y &= 0 \\ \partial_y B_z &= \mu_0 J_x \\ -\partial_x B_z &= \mu_0 J_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z. \end{aligned}$$

For the windowframe design, we will solve the middle two equations by assuming there is only transverse field ($B_z = 0$) and longitudinal current ($J_x = J_y = 0$). This leaves:

$$\begin{aligned} \partial_x B_x + \partial_y B_y &= 0 \\ \partial_x B_y - \partial_y B_x &= \mu_0 J_z. \end{aligned}$$

3 Solutions with Separate x, y Dependencies

One way of satisfying $\partial_x B_x + \partial_y B_y = 0$ is to say B_x is only a function of y and B_y is only a function of x, making both derivatives zero. That leaves the other equation

$$\partial_x B_y(x) - \partial_y B_x(y) = B'_y(x) - B'_x(y) = \mu_0 J_z.$$

If we only consider special current distributions that can be expressed in the form $J_z(x, y) = f(x) + g(y)$, then separating the x and y dependencies gives that

$$\begin{aligned} B'_y(x) &= \mu_0 f(x) \\ -B'_x(y) &= \mu_0 g(y) \end{aligned}$$

is sufficient for a solution. Or more explicitly,

$$\begin{aligned} B_y(x) &= \mu_0 \int^x f(x_1)\,\mathrm{d}x_1 \\ B_x(y) &= -\mu_0 \int^y g(y_1)\,\mathrm{d}y_1. \end{aligned}$$

The condition that the field be perpendicular to the iron surfaces sets the constant of integration to give

$$B_y(\pm R_x) = B_x(\pm R_y) = 0,$$

where R_x and R_y are the half-width and half-height of the iron aperture respectively.

4 Dipole Field

Assume there are coils of thickness X and longitudinal current density $\pm J$ on the left and right-hand sides of the windowframe. Thus $f(x) = J$ when $x \in [-R_x, -R_x + X]$ and $f(x) = -J$ when $x \in [R_x - X, R_x]$. The perfect reflection of the iron makes the images of these currents to extend to infinity vertically, so this representation of the current is valid with $g(y) = 0$. The integral for B_y trivially gives

$$B_y = \mu_0 J X$$

for $x \in [-R_x + X, R_x - X]$, i.e. the aperture of the windowframe.

Realistically, with four rectangular coils packed into a square windowframe, the coils do not reach all the way to the top and bottom. Instead, the coils are of half-height $R_x - X$, giving a filling factor $(R_x - X)/R_x$. As a crude but often quite accurate approximation, this filling factor may be multiplied to give a more realistic dipole field:

$$B_y = \frac{\mu_0 J X (R_x - X)}{R_x}.$$

5 Quadrupole Gradient

We have current density of J to the left and right, so $f(x) = J$ for $|x| \geq R_x - X$ but an opposite current of $-J$ on the top and bottom, so $g(y) = -J$ for $|y| \geq R_y - Y$. The x integral gives that $B_y(R_x) = B_y(-R_x) + 2\mu_0 J X$, which is a problem since both $B_y(R_x)$ and $B_y(-R_x)$ should be zero by the iron boundary condition. However, there is a modification to the solution that still works:

$$
\begin{aligned}
B_y(x) &= \mu_0 \int^x f(x_1)\,\mathrm{d}x_1 + Gx \\
B_x(y) &= -\mu_0 \int^y g(y_1)\,\mathrm{d}y_1 + Gy.
\end{aligned}
$$

The additional terms satisfy $\partial_x B_x + \partial_y B_y = 0$ trivially, since the functional dependencies are still intact and $\partial_x B_y - \partial_y B_x = 0$ because G is the same in both equations. This has added the field of a quadrupole gradient G onto the solution.

To cancel our previous problem, we need $2R_x G = -2\mu_0 JX$, so $G = -\mu_0 JX/R_x$. In the vertical direction, we require

$$
\begin{aligned}
-2\mu_0 JY &= 2R_y G \\
-2\mu_0 JY &= 2R_y(-\mu_0 JX/R_x) \\
Y &= R_y(X/R_x),
\end{aligned}
$$

so the coil thicknesses must be in proportion to the aspect ratio of the iron. For a square frame where $R_x = R_y$, we must have $Y = X$ i.e. equal thickness coils.

In any case, the integral contributes zero within the bore of the windowframe, leaving a field $B_y = Gx$, $B_x = Gy$, which is a quadrupole of gradient

$$
G = -\frac{\mu_0 JX}{R_x} = -\frac{\mu_0 JY}{R_y}.
$$

For a square frame of full bore $B = 2(R_x - X)$, we have $R_x = B/2 + X$, so

$$
G = -\frac{\mu_0 JX}{B/2 + X} = -\frac{2\mu_0 JX}{B + 2X} = -\frac{2\mu_0 JX}{D},
$$

where $D = 2R_x$ is the diameter of the iron frame. The gradient will never have magnitude larger than $\mu_0 J$, regardless of bore or coil thickness. This construction assumes rectangular coil cross-sections with empty corners, so it may be possible to exceed this limit by filling in the corners.