# Integrated Field of a Finite Wire

Stephen Brooks

January 29, 2019

## 1 Assumptions

The wire segment in question travels in a straight line from position $\mathbf{a}$ to $\mathbf{b}$ and carries current $I$ in the direction towards $\mathbf{b}$. The magnetic field of this segment alone will not be Maxwellian because the current does not satisfy the continuity equation. However, the field sum of a loop of such wires will be. The integration will be performed along the entire $z$ axis ($-\infty$ to $\infty$).

## 2 Derivation

The Biot-Savart law is

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \mathbf{r}}{|\mathbf{r}|^3} \, \mathrm{d}s,$$

where $\mathbf{r}$ is the vector to $\mathbf{x}$ from the relevant point on the conductor. The parametrisation

$$\mathbf{r} = \mathbf{x} - (\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})), \qquad \mathrm{d}s = |\mathbf{b} - \mathbf{a}| \, \mathrm{d}\lambda, \qquad \mathbf{I} = \frac{I(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|},$$

for $\lambda \in [0, 1]$, is used for the wire segment. $\mathbf{I}$ is constant with $s$ so the cross product can be taken outside the integral. Integrating along the $z$ axis gives

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \, \mathrm{d}z = \frac{\mu_0}{4\pi} \mathbf{I} \times \int \int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^3} \, \mathrm{d}z \, \mathrm{d}s,$$

where $z$ is the third component of $\mathbf{x}$ or of $\mathbf{r}$, equivalently, because those two vectors are related by a translation and the whole $z$ axis is integrated over in both cases.

$$\int_{-\infty}^{\infty} \frac{\mathbf{r}}{|\mathbf{r}|^3} \, \mathrm{d}z = \mathbf{f}(r_x, r_y) \qquad \text{where} \qquad \mathbf{f}(x, y) = \int_{-\infty}^{\infty} \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \, \mathrm{d}z.$$

Since $\int_{-\infty}^{\infty} \frac{1}{(k^2+z^2)^{3/2}} \, \mathrm{d}z = \frac{2}{k^2}$ and $\int_{-\infty}^{\infty} \frac{z}{(k^2+z^2)^{3/2}} \, \mathrm{d}z = 0$,

$$\mathbf{f}(x, y) = \frac{2(x, y, 0)}{x^2 + y^2}$$

and thus

$$\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{x}) \, \mathrm{d}z = \frac{\mu_0}{4\pi} \mathbf{I} \times \int \frac{2(r_x, r_y, 0)}{r_x^2 + r_y^2} \, \mathrm{d}s$$

$$
\begin{aligned}
&= \frac{\mu_0\,|\mathbf{b}-\mathbf{a}|}{2\pi}\mathbf{I}\times\int_0^1 \frac{(r_x,r_y,0)}{r_x^2+r_y^2}\,\mathrm{d}\lambda \\
&= \frac{\mu_0 I}{2\pi}(\mathbf{b}-\mathbf{a})\times\int_0^1 \frac{(r_x,r_y,0)}{r_x^2+r_y^2}\,\mathrm{d}\lambda \\
&= \frac{\mu_0 I}{2\pi}(\mathbf{b}-\mathbf{a})\times(i_x,i_y,0),
\end{aligned}
$$

where $i_x = \int_0^1 \frac{r_x}{r_x^2+r_y^2}\,\mathrm{d}\lambda$ and $i_y$ is similar with $x$ and $y$ swapped. Letting $\ell = \mathbf{b}-\mathbf{a}$ and $\mathbf{X} = \mathbf{x}-\mathbf{a}$ so that $\mathbf{r} = \mathbf{X} - \lambda\ell$ gives

$$
i_x = \int_0^1 \frac{X-\lambda l_x}{(X-\lambda l_x)^2+(Y-\lambda l_y)^2}\,\mathrm{d}\lambda.
$$

This integral is of the form

$$
\int_0^1 \frac{px+q}{(px+q)^2+(rx+s)^2}\,\mathrm{d}x
$$

$$
= \left[\frac{p\ln((px+q)^2+(rx+s)^2)+2r\arctan\frac{x(p^2+r^2)+pq+rs}{qr-ps}}{2(p^2+r^2)}\right]_0^{x=1}
$$

$$
\begin{aligned}
= \;& \frac{1}{2(p^2+r^2)}\left(p\ln((p+q)^2+(r+s)^2)+2r\arctan\frac{p^2+r^2+pq+rs}{qr-ps}-p\ln(q^2+s^2)\right. \\
& \left.-2r\arctan\frac{pq+rs}{qr-ps}\right),
\end{aligned}
$$

with the substitutions $p=-l_x=a_x-b_x$, $q=X=x-a_x$, $r=-l_y=a_y-b_y$ and $s=Y=y-a_y$. For $i_y$, swap $p$ with $r$ and $q$ with $s$. Finally, evaluate

$$
\int_{-\infty}^{\infty}\mathbf{B}(\mathbf{x})\,\mathrm{d}z = \frac{\mu_0 I}{2\pi}\ell\times(i_x,i_y,0) = \frac{\mu_0 I}{2\pi}(-l_z i_y, l_z i_x, l_x i_y - l_y i_x).
$$

## 2.1 Special case: wire in Z direction

The final formula for $i_x$ in the last section divides by zero if $l_x = l_y = 0$ ($p = r = 0$). In this case, the original expression for $i_x$ becomes simply

$$
i_x = \int_0^1 \frac{X}{X^2+Y^2}\,\mathrm{d}\lambda = \frac{X}{X^2+Y^2},
$$

so that $i_x = \frac{X}{X^2+Y^2}$ and $i_y = \frac{Y}{X^2+Y^2}$. The integrated field formula simplifies to

$$
\int_{-\infty}^{\infty}\mathbf{B}(\mathbf{x})\,\mathrm{d}z = \frac{\mu_0 I}{2\pi}(-l_z i_y, l_z i_x, l_x i_y - l_y i_x) = \frac{\mu_0 I l_z}{2\pi(X^2+Y^2)}(-Y,X,0).
$$