1 Assumptions

The wire segment in question travels in a straight line from position $a$ to $b$ and carries current $I$ in the direction towards $b$. The magnetic field of this segment alone will not be Maxwellian because the current does not satisfy the continuity equation. However, the field sum of a loop of such wires will be. The integration will be performed along the entire $z$ axis ($-\infty$ to $\infty$).

2 Derivation

The Biot-Savart law is

$$B(x) = \frac{\mu_0}{4\pi} \int \frac{I \times r}{|r|^3} \, ds,$$

where $r$ is the vector to $x$ from the relevant point on the conductor. The parametrisation

$$r = x - (a + \lambda(b - a)), \quad ds = |b - a| \, d\lambda, \quad I = \frac{I(b - a)}{|b - a|},$$

for $\lambda \in [0, 1]$, is used for the wire segment. $I$ is constant with $s$ so the cross product can be taken outside the integral. Integrating along the $z$ axis gives

$$\int_{-\infty}^{\infty} B(x) \, dz = \frac{\mu_0}{4\pi} I \times \int \int_{-\infty}^{\infty} \frac{r}{|r|^3} \, dz \, ds,$$

where $z$ is the third component of $x$ or of $r$, equivalently, because those two vectors are related by a translation and the whole $z$ axis is integrated over in both cases.

$$\int_{-\infty}^{\infty} \frac{r}{|r|^3} \, dz = f(r_x, r_y) \quad \text{where} \quad f(x, y) = \int_{-\infty}^{\infty} \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \, dz.$$

Since $\int_{-\infty}^{\infty} \frac{1}{(k^2 + z^2)^{3/2}} \, dz = \frac{2}{k^2}$ and $\int_{-\infty}^{\infty} \frac{z}{(k^2 + z^2)^{3/2}} \, dz = 0,$

$$f(x, y) = \frac{2(x, y, 0)}{x^2 + y^2}$$

and thus

$$\int_{-\infty}^{\infty} B(x) \, dz = \frac{\mu_0}{4\pi} I \times \int \frac{2(r_x, r_y, 0)}{r_x^2 + r_y^2} \, ds.$$
\[ I = \frac{\mu_0 |b-a|}{2\pi} \times \int_0^1 \left( \frac{r_x, r_y, 0}{r_x^2 + r_y^2} \right) d\lambda \]

\[ I = \frac{\mu_0 I}{2\pi} (b-a) \times \int_0^1 \left( \frac{r_x, r_y, 0}{r_x^2 + r_y^2} \right) d\lambda \]

\[ I = \frac{\mu_0 I}{2\pi} (b-a) \times (i_x, i_y, 0), \]

where \( i_x = \int_0^1 \frac{r_x}{r_x^2 + r_y^2} d\lambda \) and \( i_y \) is similar with \( x \) and \( y \) swapped. Letting \( \ell = b-a \) and \( X = x-a \) so that \( r = X - \lambda \ell \) gives

\[ i_x = \int_0^1 \frac{X - \lambda l_x}{(X - \lambda l_x)^2 + (Y - \lambda l_y)^2} d\lambda. \]

This integral is of the form

\[ \int_0^1 \frac{px + q}{(px + q)^2 + (rx + s)^2} dx = \left[ \frac{p \ln((px + q)^2 + (rx + s)^2) + 2r \arctan\left(\frac{x(p^2 + r^2) + pq + rs}{qr - ps}\right)}{2(p^2 + r^2)} \right]_{x=0}^{x=1} \]

\[ = \frac{1}{2(p^2 + r^2)} \left( p \ln((p + q)^2 + (r + s)^2) + 2r \arctan\left(\frac{p^2 + r^2 + pq + rs}{qr - ps}\right) - p \ln(q^2 + s^2) \right) \]

\[ - 2r \arctan\left(\frac{pq + rs}{qr - ps}\right), \]

with the substitutions \( p = -l_x = a_x - b_x, q = X = x - a_x, r = -l_y = a_y - b_y \) and \( s = Y = y - a_y \).

For \( i_y \), swap \( p \) with \( r \) and \( q \) with \( s \). Finally, evaluate

\[ \int_{-\infty}^{\infty} B(x) \, dz = \frac{\mu_0 I}{2\pi} \ell \times (i_x, i_y, 0) = \frac{\mu_0 I}{2\pi} (-l_z i_y, l_z i_x, l_x i_y - l_y i_x). \]

### 2.1 Special case: wire in Z direction

The final formula for \( i_x \) in the last section divides by zero if \( l_x = l_y = 0 \) (\( p = r = 0 \)). In this case, the original expression for \( i_x \) becomes simply

\[ i_x = \int_0^1 \frac{X}{X^2 + Y^2} d\lambda = \frac{X}{X^2 + Y^2}, \]

so that \( i_x = \frac{X}{X^2 + Y^2} \) and \( i_y = \frac{Y}{X^2 + Y^2} \). The integrated field formula simplifies to

\[ \int_{-\infty}^{\infty} B(x) \, dz = \frac{\mu_0 I}{2\pi} (-l_z i_y, l_z i_x, l_x i_y - l_y i_x) = \frac{\mu_0 I l_z}{2\pi (X^2 + Y^2)} (-Y, X, 0). \]