Integrals of Distorted Gaussian Functions

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1 Integral of Gaussian times Polynomial

Series solutions involving Gaussian-smoothed sources may involve integrating a slightly-distorted Gaussian function, which can be expressed as a Gaussian multiplied by a polynomial or series. The 1D Gaussian function centred at $x = 0$ is $f_{1,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ and obeys $\int_{-\infty}^{\infty} f_{1,\sigma}(x)\,\mathrm{d}x = 1$. First, calculate some derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{1,\sigma} = -\frac{1}{\sigma^2} x f_{1,\sigma} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}[x^n f_{1,\sigma}] = n x^{n-1} f_{1,\sigma} - \frac{1}{\sigma^2} x^{n+1} f_{1,\sigma}.$$

Let

$$I_n = \int_{-\infty}^{\infty} x^n f_{1,\sigma}(x)\,\mathrm{d}x,$$

so that $I_0 = 1$. Integrating both sides of the formula for $\frac{\mathrm{d}}{\mathrm{d}x}[x^n f_{1,\sigma}]$ gives

$$[x^n f_{1,\sigma}]_{-\infty}^{\infty} = n I_{n-1} - \frac{1}{\sigma^2} I_{n+1}.$$

The left hand side is zero because a Gaussian decreases faster than any polynomial can increase as $x \to \pm\infty$. The $n = 0$ case gives $I_1 = 0$, which can also be seen because x^1 is an odd function. The $n + 1$ cases give a recurrence

$$I_{n+2} = \sigma^2 (n+1) I_n,$$

which can be solved for the even numbers to give the general solution

$$I_{2n} = \frac{(2n)!}{n! 2^n} \sigma^{2n}, \qquad I_{2n+1} = 0.$$

For the 3D spherical Gaussian $f_{3,\sigma}(\mathbf{x}) = \frac{1}{(\sigma\sqrt{2\pi})^3} e^{-|\mathbf{x}|^2/2\sigma^2}$, the relationship $f_{3,\sigma}(x, y, z) = f_{1,\sigma}(x) f_{1,\sigma}(y) f_{1,\sigma}(z)$ can be used to evaluate it multiplied by 3-variable polynomial terms:

$$\int x^i y^j z^k f_{3,\sigma}(\mathbf{x})\,\mathrm{d}^3\mathbf{x} = \int_{-\infty}^{\infty} x^i f_{1,\sigma}(x)\,\mathrm{d}x \int_{-\infty}^{\infty} y^j f_{1,\sigma}(y)\,\mathrm{d}y \int_{-\infty}^{\infty} z^k f_{1,\sigma}(z)\,\mathrm{d}z = I_i I_j I_k.$$

This is only nonzero when all of i, j, k are even.

2 Integral of Gaussian of Another Function

If the argument of the Gaussian function is another function of position, integration by substitution may be used:

$$\int_{-\infty}^{\infty} f_{1,\sigma}(h(x))\,\mathrm{d}x = \int_{-\infty}^{\infty} f_{1,\sigma}(h)\frac{\mathrm{d}x}{\mathrm{d}h}\,\mathrm{d}h,$$

which works best if $h(x)$ is invertible, otherwise multiple solutions for h will have to be summed to cover all x in the original integral. As the Gaussian $f_{1,\sigma}$ peaks near $h = 0$, a series expansion of $\frac{\mathrm{d}x}{\mathrm{d}h}$ may be used, which makes each term an integral of the form in the previous section.

More explicitly, suppose $h(x)$ is a bijection and $h(x_0) = 0$. The Taylor expansion of h is

$$h(x) = h(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} h^{(n)}(x_0)(x - x_0)^n.$$

Since the first nonzero term is in $(x - x_0)^1$, the p^{th} power of $h(x)$ will have a series starting at $(x - x_0)^p$ for some coefficients:

$$h(x)^p = \sum_{n=p}^{\infty} c_{pn}(x - x_0)^n, \qquad \text{where} \qquad c_{1n} = \frac{1}{n!} h^{(n)}(x_0), \qquad c_{p+1,n} = \sum_{i=p}^{n-1} c_{pi} c_{1,n-i}.$$

This helps construct the inverse of this series that satsifies

$$x - x_0 = \sum_{n=1}^{\infty} a_n h(x)^n = \sum_{n=1}^{\infty} a_n \sum_{i=n}^{\infty} c_{ni}(x - x_0)^i = \sum_{i=1}^{\infty}\left(\sum_{n=1}^{i} a_n c_{ni}\right)(x - x_0)^i.$$

The coefficients a_n may be found by 'long division', where each successive a_n is chosen to make the coefficient of $(x - x_0)^n$ correct, starting with

$$a_1 c_{11} = 1 \qquad \Rightarrow \qquad a_1 = \frac{1}{c_{11}}$$

and continuing for $n > 1$

$$\sum_{i=1}^{n} a_i c_{in} = 0 \qquad \Rightarrow \qquad a_n = \frac{-\sum_{i=1}^{n-1} a_i c_{in}}{c_{nn}}.$$

Taking the derivative of the resulting series with respect to $h(x)$ gives

$$\frac{\mathrm{d}x}{\mathrm{d}h} = \sum_{n=1}^{\infty} n a_n h^{n-1}$$

and now the integral may be evaluated using results from the previous section:

$$\int_{-\infty}^{\infty} f_{1,\sigma}(h(x))\,\mathrm{d}x = \int_{-\infty}^{\infty} f_{1,\sigma}(h)\sum_{n=1}^{\infty} n a_n h^{n-1}\,\mathrm{d}h = \sum_{n=1}^{\infty} n a_n I_{n-1}.$$