

The Connection of Adiabaticity to the Fourier Transform

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This is probably the first thing I'd suggest anyone learns about adiabatic transitions because it tells you how a linear system reacts to an arbitrary forcing function. Not all systems are linear but they are usually locally linear near their stable fixed point. I think Scott Berg pointed out the relation to the Fourier transform to me and there's probably literature about this under generic dynamical systems.

Suppose you normalise your system so that linearly, it is the "circle problem" in complex numbers:

$$\frac{dz}{dt} = i\omega z.$$

This has solutions of the form $z(t) = Ae^{i\omega t}$ for any constant A .

Now consider where you have displaced the string of the pendulum, or the RF bucket or whatever you're trying to move adiabatically, by a time-dependent function $f(t)$. The new equation of motion is:

$$\frac{dz}{dt} = i\omega(z - f(t)).$$

1 Movement Relative to Origin

Try a solution of the form $z(t) = A(t)e^{i\omega t}$. The time derivative is:

$$\frac{dz}{dt} = \left(\frac{dA}{dt} + i\omega A \right) e^{i\omega t}.$$

Then the original equation becomes:

$$\left(\frac{dA}{dt} + i\omega A \right) e^{i\omega t} = i\omega(Ae^{i\omega t} - f(t)).$$

You can subtract $i\omega Ae^{i\omega t}$ both sides to get:

$$\frac{dA}{dt} e^{i\omega t} = i\omega(-f(t)).$$

Divide by $e^{i\omega t}$:

$$\frac{dA}{dt} = -i\omega f(t)e^{-i\omega t}.$$

Now integrate from $t = 0$ to T , assuming we start with $z(0) = A(0) = 0$:

$$\begin{aligned} A(T) &= -i\omega \int_0^T f(t)e^{-i\omega t} dt \\ &= -i\omega \mathcal{F}\{f(t)|_{[0,T]}\}(\omega). \end{aligned}$$

Thus, the amplitude you end up with is proportional to the Fourier transform of the forcing/driving function $f(t)$, evaluated at frequency ω . You may also consider a non-interacting ensemble with a whole range of ω and as long as the Fourier transform is small over that range, the final amplitude of all “particles” will be small.

In the case of an RF bucket, the independent variable z is not a particle position but a beam centroid, or 2nd moment of the beam in the bucket (or even a higher moment provided you give it the right complex phase).

2 Movement Relative to Moving Point

The $A(t)$ in the last section denoted amplitude from the origin rather than the adiabatically-moving centre point. We can redefine $z(t)$ as

$$z(t) = f(t) + A(t)e^{i\omega t}$$

to make A measure amplitude from $f(t)$. The time derivative is

$$\frac{dz}{dt} = \frac{df}{dt} + \left(\frac{dA}{dt} + i\omega A \right) e^{i\omega t},$$

which can be substituted into the equation of motion:

$$\begin{aligned} \frac{df}{dt} + \left(\frac{dA}{dt} + i\omega A \right) e^{i\omega t} &= i\omega(Ae^{i\omega t}) \\ \frac{df}{dt} + \frac{dA}{dt} e^{i\omega t} &= 0 \\ \frac{dA}{dt} &= -\frac{df}{dt} e^{-i\omega t} \\ A(T) &= -\int_0^T \frac{df(t)}{dt} e^{-i\omega t} dt \\ &= -\mathcal{F} \left\{ \frac{df(t)}{dt} \Big|_{[0,T]} \right\} (\omega). \end{aligned}$$

This is subtly different than the previous case, which it would be equal to without the restriction of the function to $[0, T]$.

3 Linear Ramp

The above formula may be applied to the case of a linear ramp from $f(0) = 0$ to $f(T) = X$. On the region $t \in [0, T]$, $f(t) = Xt/T$ and the final amplitude is:

$$\begin{aligned} A(T) &= -\int_0^T \frac{df(t)}{dt} e^{-i\omega t} dt \\ &= -\int_0^T \frac{X}{T} e^{-i\omega t} dt \\ &= -\frac{X}{T} \left[\frac{1}{-i\omega} e^{-i\omega t} \right]_{t=0}^T \\ &= \frac{X}{i\omega T} (e^{-i\omega T} - 1). \end{aligned}$$

This is proportional to the total distance X , as expected, and inversely proportional to the time taken ωT expressed in terms of radians of the periodic cycle. The dimensionless factor $e^{-i\omega T} - 1$ depends on the total phase ωT in an oscillating way and changes in magnitude from 0 to 2 during this cycle. Thus, for a single frequency system, it is possible to choose a lucky value of $T = \frac{2\pi n}{\omega}$ that precisely cancels any perturbation while not necessarily being as slow as the large T ‘adiabatic’ limit.

For systems with frequency changes, there is an adiabaticity parameter defined as

$$\epsilon = \frac{1}{\omega^2} \left| \frac{d\omega}{dt} \right|,$$

which may be made to agree with the above formula if X is interpreted as a change in *relative* frequency, i.e. $X = \log \omega$. In this case, ignoring the oscillating phase part of the above expression,

$$|A(T)| \sim \frac{X}{\omega T} = \frac{1}{\omega} \frac{dX}{dt} = \frac{1}{\omega} \frac{d \log \omega}{dt} = \frac{1}{\omega^2} \frac{d\omega}{dt} \sim \epsilon.$$

Linear ramps are often not the best choice because smoother functions can have smaller Fourier components in the large T limit. For example if $f(t)$ was the integral of a Gaussian, then the Fourier transform of $e^{-\frac{1}{2}(t/\sigma)^2}$ would be taken, giving $e^{-\frac{1}{2}(\omega\sigma)^2}$, which decreases extremely rapidly as σ (the time scale) increases.

4 Second-Order Real System

Differentiating the complex problem again and taking the real part gives:

$$\frac{dz}{dt} = i\omega z \quad \Rightarrow \quad \frac{d^2z}{dt^2} = -\omega^2 z \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\omega^2 x,$$

which is familiar simple harmonic motion, where we have defined $x = \text{Re } z$. The displaced problem

$$\frac{dz}{dt} = i\omega(z - f(t))$$

can be differentiated again to give

$$\begin{aligned} \frac{d^2z}{dt^2} &= i\omega \left(\frac{dz}{dt} - \frac{df}{dt} \right) \\ &= i\omega \left(i\omega(z - f(t)) - \frac{df}{dt} \right) \\ &= -\omega^2(z - f(t)) - i\omega \frac{df}{dt}. \end{aligned}$$

Now assuming $f(t)$ is always real (and ω), taking the real part gives

$$\frac{d^2x}{dt^2} = -\omega^2(x - f(t)),$$

which is simple harmonic motion with a variable displacement, as required.

The description of the state of this second-order system requires a second variable, one choice is the velocity:

$$v = \frac{dx}{dt} = \text{Re} \frac{dz}{dt} = \text{Re}(i\omega z) = -\omega \text{Im } z,$$

implying the complex variable is given as $z = x - i\frac{v}{\omega}$ in terms of the real ones.