Off-Axis RF Fields Extrapolated from On-Axis Values

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1 Maxwell's Equations for an RF Field

Start with Maxwell's equations

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right),\end{aligned}$$

set the sources $\rho, \mathbf{J}$ to zero and look for a periodic solution with $\mathbf{E} = \mathbf{E}_0 \cos \omega t$, $\mathbf{B} = \mathbf{B}_0 \sin \omega t$ where $\mathbf{E}_0, \mathbf{B}_0$ do not depend on time. This gives

$$\begin{aligned} & & \nabla \cdot \mathbf{E}_0 &= 0 \\ \nabla \times \mathbf{E}_0 \cos \omega t = -\frac{\partial \mathbf{B}}{\partial t} = -\omega \mathbf{B}_0 \cos \omega t \quad &\Rightarrow \quad \nabla \times \mathbf{E}_0 &= -\omega \mathbf{B}_0 \\ & & \nabla \cdot \mathbf{B}_0 &= 0 \\ \nabla \times \mathbf{B}_0 \sin \omega t = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2}(-\omega \mathbf{E}_0 \sin \omega t) \quad &\Rightarrow \quad \nabla \times \mathbf{B}_0 &= -\frac{\omega}{c^2} \mathbf{E}_0.\end{aligned}$$

Taking the curl of the second equation, notice that

$$\nabla \times (\nabla \times \mathbf{E}_0) = -\omega \nabla \times \mathbf{B}_0 = \frac{\omega^2}{c^2} \mathbf{E}_0.$$

It turns out that finding an $\mathbf{E}_0$ that solves this equation is sufficient to satisfy all four, because $\mathbf{B}_0$ can be defined as $-\frac{1}{\omega} \nabla \times \mathbf{E}_0$ and then both $\mathbf{E}_0, \mathbf{B}_0$ are expressed as the curl of another vector field, making their divergences zero, satisfying the other two equations.

2 Case with No Angular Dependency

The aim is to build a solution for $\mathbf{E}_0$ out of a sum of simpler functions. Defining $r = \sqrt{x^2 + y^2}$, consider a longitudinal (z directed) vector field $\mathbf{a}$ with r^k scaling in the radial direction:

$$a_z = r^k f(z), \quad a_x = a_y = 0.$$

The curl of this has two nonzero components:

$$\begin{aligned}(\nabla \times \mathbf{a})_x &= \partial_y a_z = k r^{k-1} \frac{\partial r}{\partial y} f(z) = k r^{k-2} y f(z), \\ (\nabla \times \mathbf{a})_y &= -\partial_x a_z = -k r^{k-1} \frac{\partial r}{\partial x} f(z) = -k r^{k-2} x f(z).\end{aligned}$$

Now consider a similar term $\mathbf{b}$ that points azimuthally around the axis:

$$b_x = r^{k-1} y f(z), \quad b_y = -r^{k-1} x f(z), \quad b_z = 0.$$

Its curl components are:

$$\begin{aligned}(\nabla \times \mathbf{b})_x &= -\partial_z b_y = r^{k-1} x f'(z), \\ (\nabla \times \mathbf{b})_y &= \partial_z b_x = r^{k-1} y f'(z), \\ (\nabla \times \mathbf{b})_z &= \partial_x b_y - \partial_y b_x \\ &= -r^{k-1} f(z) - (k-1) r^{k-3} x^2 f(z) - r^{k-1} f(z) - (k-1) r^{k-3} y^2 f(z) \\ &= -2 r^{k-1} f(z) - (k-1) r^{k-1} f(z) \\ &= -(k+1) r^{k-1} f(z).\end{aligned}$$

Finally, define a radial term $\mathbf{c}$:

$$c_x = r^{k-1} x f(z), \quad c_y = r^{k-1} y f(z), \quad c_z = 0,$$

that has curl components

$$\begin{aligned}(\nabla \times \mathbf{c})_x &= -\partial_z c_y = -r^{k-1} y f'(z), \\ (\nabla \times \mathbf{c})_y &= \partial_z c_x = r^{k-1} x f'(z), \\ (\nabla \times \mathbf{c})_z &= \partial_x c_y - \partial_y c_x \\ &= (k-1) r^{k-3} x y f(z) - (k-1) r^{k-3} x y f(z) \\ &= 0.\end{aligned}$$

As a shorthand, subscripts will indicate substitutions of k or f in these terms, omitting any that are unchanged. In this notation,

$$\begin{aligned}\nabla \times \mathbf{a} &= k \mathbf{b}_{k-1}, \\ \nabla \times \mathbf{b} &= -(k+1) \mathbf{a}_{k-1} + \mathbf{c}_{f'}, \\ \nabla \times \mathbf{c} &= -\mathbf{b}_{f'}.\end{aligned}$$

Now the double curls may be evaluated:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{a}) &= k \left(-k \mathbf{a}_{k-2} + \mathbf{c}_{k-1,f'} \right) \\ &= -k^2 \mathbf{a}_{k-2} + k \mathbf{c}_{k-1,f'}, \\ \nabla \times (\nabla \times \mathbf{b}) &= -(k+1) \left((k-1) \mathbf{b}_{k-2} \right) + \left(-\mathbf{b}_{f''} \right) \\ &= -(k^2-1) \mathbf{b}_{k-2} - \mathbf{b}_{f''}, \\ \nabla \times (\nabla \times \mathbf{c}) &= -\left(-(k+1) \mathbf{a}_{k-1,f'} + \mathbf{c}_{f''} \right) \\ &= (k+1) \mathbf{a}_{k-1,f'} - \mathbf{c}_{f''}.\end{aligned}$$

Analogously with the magnetic field solutions in [1], seek a solution of the form

$$\mathbf{E}_0 = \sum_{k=0}^{\infty} \mathbf{a}_{k,f_k} + \mathbf{b}_{k,g_k} + \mathbf{c}_{k,h_k}$$

for some functions f_k, g_k, h_k. The double curl is

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}_0) &= \sum_{k=0}^{\infty} -k^2 \mathbf{a}_{k-2,f_k} + k\mathbf{c}_{k-1,f_k'} \\ &- (k^2-1)\mathbf{b}_{k-2,g_k} - \mathbf{b}_{g_k''} \\ &+ (k+1)\mathbf{a}_{k-1,h_k'} - \mathbf{c}_{h_k''}.\end{aligned}$$

Equating this to $\frac{\omega^2}{c^2}\mathbf{E}_0$ and only taking the order k terms gives

$$\begin{aligned}\frac{\omega^2}{c^2}(\mathbf{a}_{f_k} + \mathbf{b}_{g_k} + \mathbf{c}_{h_k}) &= -(k+2)^2 \mathbf{a}_{f_{k+2}} + (k+1)\mathbf{c}_{f_{k+1}'} \\ &- ((k+2)^2-1)\mathbf{b}_{g_{k+2}} - \mathbf{b}_{g_k''} \\ &+ (k+2)\mathbf{a}_{h_{k+1}'} - \mathbf{c}_{h_k''}.\end{aligned}$$

Equating the longitudinal, azimuthal and radial vector coefficients gives

$$\begin{aligned}\frac{\omega^2}{c^2} f_k &= -(k+2)^2 f_{k+2} + (k+2)h_{k+1}', \\ \frac{\omega^2}{c^2} g_k &= -((k+2)^2-1)g_{k+2} - g_k'', \\ \frac{\omega^2}{c^2} h_k &= (k+1)f_{k+1}' - h_k''.\end{aligned}$$

At this point, the first two equations can be rearranged to give f_{k+2} and g_{k+2} in terms of lower-numbered functions, as required to calculate them all via a recurrence. However, the highest function in the third equation is f_{k+1}' and not h. Fortunately, some careful cancellation can get rid of both the f_{k+1}' and h_k'' terms.

Taking the first equation for the $k-1$ case and taking the z derivative gives

$$\frac{\omega^2}{c^2} f_{k-1}' = -(k+1)^2 f_{k+1}' + (k+1)h_k''.$$

Multiplying the third equation by $-(k+1)$ gives

$$-(k+1)\frac{\omega^2}{c^2} h_k = -(k+1)^2 f_{k+1}' + (k+1)h_k''.$$

The right hand sides are now identical, implying that

$$f_{k-1}' = -(k+1)h_k,$$

which allows h to be eliminated entirely. In particular,

$$f_k'' = -(k+2)h_{k+1}',$$

which allows writing the first two equations as:

$$\begin{aligned}\frac{\omega^2}{c^2} f_k &= -(k+2)^2 f_{k+2} - f_k'', \\ \frac{\omega^2}{c^2} g_k &= -((k+2)^2-1)g_{k+2} - g_k''.\end{aligned}$$

Now everything can be written as simple recurrence formulae

$$\begin{aligned} f_{k+2} &= \frac{-1}{(k+2)^2}\left(\frac{\omega^2}{c^2}f_k + f_k''\right), \\ g_{k+2} &= \frac{-1}{(k+2)^2-1}\left(\frac{\omega^2}{c^2}g_k + g_k''\right), \\ h_{k+1} &= \frac{-1}{k+2}f_k'. \end{aligned}$$

To avoid discontinuities, the lowest allowed nonzero functions are f_0 for the on-axis electric field and g_1 and h_1.

2.1 3D Field from On-Axis Electric Field

In the case where just the on-axis electric field $f_0(z) = f(z)$ is specified, azimuthal components g_k can be set to zero as their recurrence relation is independent of the rest. f_k and h_k are all linear combinations of derivatives of f. Specifically, for all k, $f_{2k+1} = 0$, $h_{2k} = 0$ and

$$\begin{aligned} f_{2k} &= \left(\prod_{j=1}^{k}\frac{-1}{(2j)^2}\right)\left(\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)^k f \\ &= \frac{(-1)^k}{4^k(k!)^2}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\omega^2}{c^2}\right)^{k-j} f^{(2j)}, \\ h_{2k+1} &= \frac{-1}{2k+2}f_{2k}' \\ &= \frac{(-1)^{k+1}}{2^{2k+1}k!(k+1)!}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\omega^2}{c^2}\right)^{k-j} f^{(2j+1)}. \end{aligned}$$

The field shape explicitly is:

$$\begin{aligned} \mathbf{E}_0 &= \sum_{k=0}^{\infty}\mathbf{a}_{k,f_k} + \mathbf{c}_{k,h_k} \\ &= \sum_{k=0}^{\infty}[r^{k-1}xh_k(z), r^{k-1}yh_k(z), r^k f_k(z)]. \end{aligned}$$

The magnetic component is:

$$\begin{aligned} \mathbf{B}_0 &= -\frac{1}{\omega}\nabla\times\mathbf{E}_0 \\ &= -\frac{1}{\omega}\sum_{k=0}^{\infty}\nabla\times\mathbf{a}_{k,f_k} + \nabla\times\mathbf{c}_{k,h_k} \\ &= -\frac{1}{\omega}\sum_{k=0}^{\infty}k\mathbf{b}_{k-1,f_k} - \mathbf{b}_{k,h_k'} \\ &= -\frac{1}{\omega}\sum_{k=0}^{\infty}(k+1)\mathbf{b}_{k,f_{k+1}} - \mathbf{b}_{k,h_k'} \\ &= -\frac{1}{\omega}\sum_{k=0}^{\infty}((k+1)f_{k+1}(z) - h_k'(z))[r^{k-1}y, -r^{k-1}x, 0]. \end{aligned}$$

The last expression can be simplified slightly using the recurrence relations for f and h:

$$\begin{aligned}(k+1)f_{k+1} &= \frac{-1}{k+1}\left(\frac{\omega^2}{c^2}f_{k-1}+f''_{k-1}\right)\\ h'_k &= \frac{-1}{k+1}f''_{k-1}\\ \Rightarrow \qquad (k+1)f_{k+1}-h'_k &= \frac{-1}{k+1}\frac{\omega^2}{c^2}f_{k-1}.\end{aligned}$$

Thus,

$$\mathbf{B}_0 = \frac{\omega}{c^2}\sum_{k=0}^{\infty}\frac{f_{k-1}(z)}{k+1}[r^{k-1}y, -r^{k-1}x, 0].$$

The full time-dependent field can be recovered with $\mathbf{E} = \mathbf{E}_0\cos\omega t$ and $\mathbf{B} = \mathbf{B}_0\sin\omega t$.

3 Case with Angular Dependency

Defining cylindrical coordinates with $x = r\cos\theta$ and $y = r\sin\theta$, consider a longitudinal (z directed) vector field $\mathbf{a}$ with n-fold rotational symmetry in θ and r^k scaling in the radial direction:

$$a_z = e^{in\theta}r^k f(z), \quad a_r = a_\theta = 0.$$

Here, the field components are complex so the cosine or sine part can be extracted by taking the real or imaginary component respectively. These components still satisfy all the differential equations. The curl in cylindrical polar coordinates (r,θ,z) has two nonzero components:

$$\begin{aligned}(\nabla\times\mathbf{a})_r &= \tfrac{1}{r}\partial_\theta a_z = ine^{in\theta}r^{k-1}f(z),\\ (\nabla\times\mathbf{a})_\theta &= -\partial_r a_z = -e^{in\theta}kr^{k-1}f(z).\end{aligned}$$

It is sufficient to consider two other single-component terms $\mathbf{b}$ and $\mathbf{c}$ where

$$b_\theta = e^{in\theta}r^k f(z), \quad b_r = b_z = 0,$$

$$c_r = e^{in\theta}r^k f(z), \quad c_\theta = c_z = 0.$$

The curl of $\mathbf{b}$ has nonzero components

$$\begin{aligned}(\nabla\times\mathbf{b})_r &= -\partial_z b_\theta = -e^{in\theta}r^k f'(z),\\ (\nabla\times\mathbf{b})_z &= \tfrac{1}{r}\partial_r(rb_\theta) = e^{in\theta}(k+1)r^{k-1}f(z).\end{aligned}$$

The curl of $\mathbf{c}$ has nonzero components

$$\begin{aligned}(\nabla\times\mathbf{c})_\theta &= \partial_z c_r = e^{in\theta}r^k f'(z),\\ (\nabla\times\mathbf{c})_z &= -\tfrac{1}{r}\partial_\theta c_r = -ine^{in\theta}r^{k-1}f(z).\end{aligned}$$

As a shorthand, subscripts will indicate substitutions of k or f in these fields, omitting any that are unchanged. In this notation,

$$\begin{aligned}\nabla\times\mathbf{a} &= in\mathbf{c}_{k-1} - k\mathbf{b}_{k-1},\\ \nabla\times\mathbf{b} &= -\mathbf{c}_{f'} + (k+1)\mathbf{a}_{k-1},\\ \nabla\times\mathbf{c} &= \mathbf{b}_{f'} - in\mathbf{a}_{k-1}.\end{aligned}$$

Now the double curls may be evaluated:

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{a}) &= in(\mathbf{b}_{k-1,f'}-in\mathbf{a}_{k-2})-k(-\mathbf{c}_{k-1,f'}+k\mathbf{a}_{k-2})\\
&= (n^2-k^2)\mathbf{a}_{k-2}+in\mathbf{b}_{k-1,f'}+k\mathbf{c}_{k-1,f'},\\
\nabla\times(\nabla\times\mathbf{b}) &= -(\mathbf{b}_{f''}-in\mathbf{a}_{k-1,f'})+(k+1)(in\mathbf{c}_{k-2}-(k-1)\mathbf{b}_{k-2})\\
&= in\mathbf{a}_{k-1,f'}-\mathbf{b}_{f''}-(k^2-1)\mathbf{b}_{k-2}+in(k+1)\mathbf{c}_{k-2},\\
\nabla\times(\nabla\times\mathbf{c}) &= (-\mathbf{c}_{f''}+(k+1)\mathbf{a}_{k-1,f'})-in(in\mathbf{c}_{k-2}-(k-1)\mathbf{b}_{k-2})\\
&= (k+1)\mathbf{a}_{k-1,f'}+in(k-1)\mathbf{b}_{k-2}-\mathbf{c}_{f''}+n^2\mathbf{c}_{k-2}.
\end{aligned}$$

Seek a solution of the form

$$\mathbf{E}_0=\sum_{k=0}^{\infty}\mathbf{a}_{k,f_k}+\mathbf{b}_{k,g_k}+\mathbf{c}_{k,h_k}$$

for some functions f_k, g_k, h_k. The double curl is

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{E}_0) &= \sum_{k=0}^{\infty}(n^2-k^2)\mathbf{a}_{k-2,f_k}+in\mathbf{b}_{k-1,f_k'}+k\mathbf{c}_{k-1,f_k'}\\
&+ in\mathbf{a}_{k-1,g_k'}-\mathbf{b}_{g_k''}-(k^2-1)\mathbf{b}_{k-2,g_k}+in(k+1)\mathbf{c}_{k-2,g_k}\\
&+ (k+1)\mathbf{a}_{k-1,h_k'}+in(k-1)\mathbf{b}_{k-2,h_k}-\mathbf{c}_{h_k''}+n^2\mathbf{c}_{k-2,h_k}.
\end{aligned}$$

Equating this to $\frac{\omega^2}{c^2}\mathbf{E}_0$ and only taking the r^k terms gives

$$\begin{aligned}
\frac{\omega^2}{c^2}(\mathbf{a}_{f_k}+\mathbf{b}_{g_k}+\mathbf{c}_{h_k}) &= (n^2-(k+2)^2)\mathbf{a}_{f_{k+2}}+in\mathbf{b}_{f_{k+1}'}+(k+1)\mathbf{c}_{f_{k+1}'}\\
&+ in\mathbf{a}_{g_{k+1}'}-\mathbf{b}_{g_k''}-((k+2)^2-1)\mathbf{b}_{g_{k+2}}+in(k+3)\mathbf{c}_{g_{k+2}}\\
&+ (k+2)\mathbf{a}_{h_{k+1}'}+in(k+1)\mathbf{b}_{h_{k+2}}-\mathbf{c}_{h_k''}+n^2\mathbf{c}_{h_{k+2}}.
\end{aligned}$$

Equating the longitudinal, azimuthal and radial vector coefficients gives

$$\begin{aligned}
\frac{\omega^2}{c^2}f_k &= (n^2-(k+2)^2)f_{k+2}+ing_{k+1}'+(k+2)h_{k+1}',\\
\frac{\omega^2}{c^2}g_k &= inf_{k+1}'-g_k''-((k+2)^2-1)g_{k+2}+in(k+1)h_{k+2},\\
\frac{\omega^2}{c^2}h_k &= (k+1)f_{k+1}'+in(k+3)g_{k+2}-h_k''+n^2h_{k+2}.
\end{aligned}$$

The highest function in the first equation is f_{k+2}, which is good for setting up a recurrence. However, this time the g and h equations have a combination of g_{k+2} and h_{k+2} in each of them and the two combinations are linearly dependent (note $(k+2)^2-1=(k+1)(k+3)$), so there is not enough information to solve for g_{k+2} and h_{k+2} separately. On the other hand, the linear dependence allows one equation to be obtained where these terms are removed entirely.

To simplify manipulation, define the mixtures

$$m_k=ing_k+(k+1)h_k \qquad \text{and} \qquad p_k=(k+1)g_k-inh_k.$$

Now the three equations can be written

$$\begin{aligned}
\frac{\omega^2}{c^2}f_k &= (n^2-(k+2)^2)f_{k+2}+m_{k+1}',\\
\frac{\omega^2}{c^2}g_k &= inf_{k+1}'-g_k''-(k+1)p_{k+2},\\
\frac{\omega^2}{c^2}h_k &= (k+1)f_{k+1}'-h_k''+inp_{k+2}.
\end{aligned}$$

Further mixing the second and third equations yields

$$\begin{aligned}\frac{\omega^2}{c^2}m_k &= ((k+1)^2-n^2)f'_{k+1} - m''_k,\\ \frac{\omega^2}{c^2}p_k &= -p''_k + (n^2-(k+1)^2)p_{k+2}.\end{aligned}$$

The p equation is now completely decoupled. It can also be seen that the coefficients of f_n, f'_n and p_{n+1} are zero in certain equations as k is varied. This typically allows arbitrary values to be set for f_n or p_{n+1} to start the recurrence.

Take the z derivative of the f equation and the $k+1$ index of the m equation to get

$$\begin{aligned}\frac{\omega^2}{c^2}f'_k &= (n^2-(k+2)^2)f'_{k+2} + m''_{k+1},\\ \frac{\omega^2}{c^2}m_{k+1} &= ((k+2)^2-n^2)f'_{k+2} - m''_{k+1}.\end{aligned}$$

This implies $m_{k+1} = -f'_k$ if $\omega \neq 0$. That enables the f and m equations to be rewritten as

$$\begin{aligned}\frac{\omega^2}{c^2}f_k &= (n^2-(k+2)^2)f_{k+2} - f''_k,\\ \frac{\omega^2}{c^2}m_k &= (n^2-(k+1)^2)m_{k+2} - m''_k.\end{aligned}$$

The m equation here is mostly redundant as it can be derived for $k \geq 1$ by differentiating the f equation and using $m_{k+1} = -f'_k$.

For $k \neq n-1$, we may recover g and h with

$$g_k = \frac{inm_k + (k+1)p_k}{(k+1)^2-n^2} \qquad \text{and} \qquad h_k = \frac{(k+1)m_k - inp_k}{(k+1)^2-n^2}$$

but the case $k = n-1$ is special because the mapping to m, p is non-invertible ($m_{n-1} = ip_{n-1}$). This means an additional variable (say h_{n-1}) and transverse equation is required at this point, for instance

$$\frac{\omega^2}{c^2}h_{n-1} = nf'_n - h''_{n-1} + inp_{n+1}.$$

The two lowest-order arbitrary functions can now be h_{n-1} and f_n, with p_{n+1} being determined by the equation above.

With h_{n-1} being set, what should the other component, say m_{n-1}, be? It turns out that m_{n-1} (and p_{n-1}) are at the top of the downward recurrence sequence and must be set to zero if lower-order terms are not wanted. Figure 1 shows the linkage of the various variables. There is an out-of-parity network of variables (f_{n+2j+1} and $(m,p)_{n+2j}$) that extends infinitely in both directions. These are set to zero because solutions that start at a given radial power are of interest. The remaining network is broken in two by the equations where a coefficient is zero: f_n and p_{n+1} are at the bottom of the upward portion and f_{n-2} and p_{n-1} at the top of the downward portion. Given that $m_{n-1} = ip_{n-1}$, a new node is added for h_{n-1} that is related to f_n and p_{n+1} by the extra transverse equation. This forms the true start of the transverse recurrence, with f_n still free to choose for the longitudinal part.

Setting $m_{n-1} = 0$ gives

$$0 = ing_{n-1} + nh_{n-1}$$

so that $g_{n-1} = ih_{n-1}$.

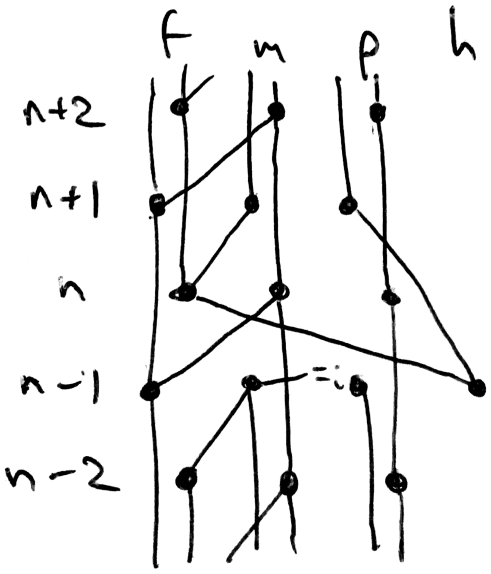

Figure 1: Linkage of recurrence variables.

Overall, starting with a lowest-order longitudinal function f_n and transverse function h_{n-1},

$$g_{n-1} = ih_{n-1}, \qquad p_{n+1} = \frac{-i}{n}\left(\frac{\omega^2}{c^2}h_{n-1} + h''_{n-1}\right) + if'_n,$$

and then for $k \geq n$,

$$\begin{aligned} f_{k+2} &= \frac{1}{n^2-(k+2)^2}\left(\frac{\omega^2}{c^2}f_k + f''_k\right), \\ m_{k+1} &= -f'_k, \\ p_{k+2} &= \frac{1}{n^2-(k+1)^2}\left(\frac{\omega^2}{c^2}p_k + p''_k\right). \end{aligned}$$

All other parity terms f_{n+2j+1}, m_{n+2j}, p_{n+2j} are set to zero.

3.1 3D Field from Longitudinal Electric Multipole

Suppose the lowest-order electric field is a longitudinal multipole of order n, so $f_n(z) = f(z)$ and $h_{n-1} = 0$. The extra transverse equation gives $p_{n+1} = if'_n$. For all $k \geq 0$, $f_{n+2k+1} = 0$, $m_{n+2k} = 0$ and

$$\begin{aligned} f_{n+2k} &= \left(\prod_{j=1}^{k}\frac{1}{n^2-(n+2j)^2}\right)\left(\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)^k f, \\ m_{n+2k+1} &= -f'_{n+2k}, \\ p_{n+2k+1} &= \left(\prod_{j=1}^{k}\frac{1}{n^2-(n+2j)^2}\right)\left(\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)^k p_{n+1} \\ &= if'_{n+2k}. \end{aligned}$$

Converting back to g and h,

$$\begin{aligned} g_{n+2k+1} &= \frac{in(-f'_{n+2k}) + (n+2k+2)(if'_{n+2k})}{(n+2k+2)^2 - n^2} \\ &= \frac{2k+2}{(n+2k+2)^2 - n^2}(if'_{n+2k}) = \frac{if'_{n+2k}}{2n+2k+2}, \\ h_{n+2k+1} &= \frac{(n+2k+2)(-f'_{n+2k}) - in(if'_{n+2k})}{(n+2k+2)^2 - n^2} \\ &= \frac{-(2k+2)}{(n+2k+2)^2 - n^2} f'_{n+2k} = \frac{-f'_{n+2k}}{2n+2k+2}. \end{aligned}$$

The field shape explicitly is:

$$\begin{aligned} \mathbf{E}_0 &= \sum_{k=n}^{\infty} \mathbf{a}_{k,f_k} + \mathbf{b}_{k,g_k} + \mathbf{c}_{k,h_k} \\ &= \sum_{k=n}^{\infty} e^{in\theta} r^k [h_k(z), g_k(z), f_k(z)]_{r\theta z}. \end{aligned}$$

State the transverse and longitudinal parts of $\mathbf{E}_0$ separately keeping only the nonzero terms:

$$\begin{aligned} \mathbf{E}_{0xy} &= \sum_{k=0}^{\infty} e^{in\theta} r^{n+2k+1} \frac{f'_{n+2k}(z)}{2n+2k+2} [-1, i]_{r\theta}, \\ E_{0z} &= \sum_{k=0}^{\infty} e^{in\theta} r^{n+2k} f_{n+2k}(z). \end{aligned}$$

Noting that $e^{in\theta} = \cos n\theta + i \sin n\theta$,

$$e^{in\theta}[-1, i]_{r\theta} = [-\cos n\theta - i \sin n\theta, -\sin n\theta + i \cos n\theta]_{r\theta},$$

which makes it easier to take the real or imaginary part of the field depending on whether the multipole is normal or skew. Another useful formula is $r[a, b]_{r\theta} = [ax - by, ay + bx]$.

The magnetic component is

$$\mathbf{B}_0 = -\frac{1}{\omega} \nabla \times \mathbf{E}_0$$

but first express $\mathbf{E}_0$ in terms of the basis functions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\mathbf{E}_0 = \sum_{k=0}^{\infty} \frac{1}{2n+2k+2} \left(i\mathbf{b}_{n+2k+1,f'_{n+2k}} - \mathbf{c}_{n+2k+1,f'_{n+2k}} \right) + \sum_{k=0}^{\infty} \mathbf{a}_{n+2k,f_{n+2k}}.$$

The longitudinal part gives

$$\nabla \times \mathbf{a}_{n+2k,f_{n+2k}} = in\mathbf{c}_{n+2k-1,f_{n+2k}} - (n+2k)\mathbf{b}_{n+2k-1,f_{n+2k}}$$

and the transverse terms give

$$\begin{aligned} \nabla \times \left(i\mathbf{b}_{n+2k+1,f'_{n+2k}} \right) &= -i\mathbf{c}_{n+2k+1,f''_{n+2k}} + i(n+2k+2)\mathbf{a}_{n+2k,f'_{n+2k}}, \\ \nabla \times \left(-\mathbf{c}_{n+2k+1,f'_{n+2k}} \right) &= in\mathbf{a}_{n+2k,f'_{n+2k}} - \mathbf{b}_{n+2k+1,f''_{n+2k}}. \end{aligned}$$

The longitudinal **B** field is found from collecting the **a** terms:

$$B_{0z} = -\frac{1}{\omega}\sum_{k=0}^{\infty} ie^{in\theta}r^{n+2k}f'_{n+2k}(z)$$

and the transverse field from the **b** and **c** terms:

$$\mathbf{B}_{0xy} = -\frac{1}{\omega}\sum_{k=0}^{\infty} e^{in\theta}r^{n+2k-1}\left(f_{n+2k}(z)[in,-(n+2k)]_{r\theta} + \frac{r^2}{2n+2k+2}f''_{n+2k}(z)[-i,-1]_{r\theta}\right).$$

For this expression,

$$\begin{aligned} e^{in\theta}[in,-(n+2k)]_{r\theta} &= [-n\sin n\theta + in\cos n\theta, -(n+2k)\cos n\theta - i(n+2k)\sin n\theta]_{r\theta}, \\ e^{in\theta}[-i,-1]_{r\theta} &= [\sin n\theta - i\cos n\theta, -\cos n\theta - i\sin n\theta]_{r\theta}. \end{aligned}$$

The full time-dependent field can be recovered with $\mathbf{E} = \mathbf{E}_0\cos\omega t$ and $\mathbf{B} = \mathbf{B}_0\sin\omega t$. However, note that the original r^n longitudinal electric multipole has produced an r^{n-1} transverse magnetic multipole via the definition of $\mathbf{B}_0$. This multipole is also multiplied by a factor of $\frac{1}{\omega}$, which will become large for low frequencies. An approach starting with a transverse electric field will therefore also be presented.

3.2 3D Field from Transverse Electric Multipole

Suppose the lowest-order electric field is a transverse multipole with order n symmetry, which actually corresponds to h_{n-1} as the h, g functions multiply r, θ basis vectors that rotate once themselves. Set $h_{n-1}(z) = h(z)$ and $f_n = 0$ to remove longitudinal field of the same order. This means that all f_k, m_k terms are zero, as well as p_{n+2k} that form an infinite two-way chain including lower-order terms. The lowest order equations give

$$g_{n-1} = ih_{n-1}, \qquad p_{n+1} = \frac{-i}{n}\left(\frac{\omega^2}{c^2}h_{n-1} + h''_{n-1}\right)$$

and the recurrence for p gives, for $k \geq 0$,

$$\begin{aligned} p_{n+2k+1} &= \left(\prod_{j=1}^{k}\frac{1}{n^2-(n+2j)^2}\right)\left(\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)^k p_{n+1} \\ &= \frac{-i}{n}\left(\prod_{j=1}^{k}\frac{1}{n^2-(n+2j)^2}\right)\left(\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)^{k+1} h. \end{aligned}$$

Converting back to g and h,

$$\begin{aligned} g_{n+2k+1} &= \frac{(n+2k+2)p_{n+2k+1}}{(n+2k+2)^2-n^2}, \\ h_{n+2k+1} &= \frac{-inp_{n+2k+1}}{(n+2k+2)^2-n^2}. \end{aligned}$$

The denominator can be merged with the product to give

$$(g,h)_{n+2k+1} = \left(\frac{i(n+2k+2)}{n},1\right)\left(\prod_{j=1}^{k+1}\frac{1}{n^2-(n+2j)^2}\right)\left(\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)^{k+1} h.$$

This is actually true for $k \geq -1$ as it also agrees with the values of g_{n-1} and (trivially) h_{n-1}.

There is no longitudinal $\mathbf{E}$ field ($f_k = 0$), so the field shape is:

$$\begin{aligned}\mathbf{E}_0 &= \sum_{k=n-1}^{\infty} e^{in\theta} r^k [h_k(z), g_k(z)]_{r\theta} \\ &= \sum_{k=-1}^{\infty} e^{in\theta} r^{n+2k+1} \left[1, \frac{i(n+2k+2)}{n}\right]_{r\theta} h_{n+2k+1}(z).\end{aligned}$$

Noting that $e^{in\theta} = \cos n\theta + i \sin n\theta$, the overall real and imaginary components are

$$e^{in\theta} \left[1, \frac{i(n+2k+2)}{n}\right]_{r\theta} = \left[\cos n\theta + i \sin n\theta, \frac{n+2k+2}{n}(-\sin n\theta + i \cos n\theta)\right]_{r\theta}.$$

The magnetic component is

$$\mathbf{B}_0 = -\frac{1}{\omega} \nabla \times \mathbf{E}_0.$$

Expressing $\mathbf{E}_0$ in terms of the transverse basis functions $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{E}_0 = \sum_{k=-1}^{\infty} \mathbf{c}_{n+2k+1, h_{n+2k+1}} + \frac{i(n+2k+2)}{n} \mathbf{b}_{n+2k+1, h_{n+2k+1}}.$$

The curls of these terms are

$$\nabla \times \mathbf{c}_{n+2k+1, h_{n+2k+1}} = \mathbf{b}_{n+2k+1, h'_{n+2k+1}} - in\mathbf{a}_{n+2k, h_{n+2k+1}},$$

$$\nabla \times \left(\frac{i(n+2k+2)}{n} \mathbf{b}_{n+2k+1, h_{n+2k+1}}\right) =$$

$$\frac{i(n+2k+2)}{n} \left(-\mathbf{c}_{n+2k+1, h'_{n+2k+1}} + (n+2k+2)\mathbf{a}_{n+2k, h_{n+2k+1}}\right).$$

The longitudinal $\mathbf{B}$ field is found from collecting the $\mathbf{a}$ terms:

$$\begin{aligned}B_{0z} &= -\frac{1}{\omega} \sum_{k=-1}^{\infty} \left(-in + \frac{i(n+2k+2)^2}{n}\right) e^{in\theta} r^{n+2k} h_{n+2k+1}(z) \\ &= -\frac{1}{\omega} \sum_{k=0}^{\infty} \frac{(n+2k+2)^2 - n^2}{n} i e^{in\theta} r^{n+2k} h_{n+2k+1}(z).\end{aligned}$$

Here, the possibility of a lower-order r^{n-2} term (h_{n-1} being r^{n-1}) is avoided because its coefficient is zero and it can be dropped from the sum. The transverse field comes from the $\mathbf{b}$ and $\mathbf{c}$ terms:

$$\mathbf{B}_{0xy} = -\frac{1}{\omega} \sum_{k=-1}^{\infty} \left[-\frac{i(n+2k+2)}{n}, 1\right]_{r\theta} e^{in\theta} r^{n+2k+1} h'_{n+2k+1}(z).$$

This is actually the same order as, and quite similar to, the expression for $\mathbf{E}_0$ (which is entirely transverse). The real and imaginary components are

$$e^{in\theta} \left[-\frac{i(n+2k+2)}{n}, 1\right]_{r\theta} = \left[\frac{n+2k+2}{n}(\sin n\theta - i \cos n\theta), \cos n\theta + i \sin n\theta\right]_{r\theta}.$$

The full time-dependent field can be recovered with $\mathbf{E} = \mathbf{E}_0 \cos \omega t$ and $\mathbf{B} = \mathbf{B}_0 \sin \omega t$.

3.3 Electromagnetic Duality

Considering the equations

$$\begin{aligned} \nabla \cdot \mathbf{E}_0 = \nabla \cdot \mathbf{B}_0 &= 0, \\ \nabla \times \mathbf{E}_0 &= -\omega \mathbf{B}_0 \quad \text{and} \\ \nabla \times \mathbf{B}_0 &= -\frac{\omega}{c^2}\mathbf{E}_0, \end{aligned}$$

it becomes clear that $\mathbf{E}_* = c\mathbf{B}_0$, $\mathbf{B}_* = \frac{1}{c}\mathbf{E}_0$ is also a solution:

$$\begin{aligned} \nabla \times \mathbf{E}_* = \nabla \times (c\mathbf{B}_0) = c\left(-\frac{\omega}{c^2}\mathbf{E}_0\right) = -\frac{\omega}{c}\mathbf{E}_0 &= -\omega \mathbf{B}_*, \\ \nabla \times \mathbf{B}_* = \nabla \times \left(\frac{1}{c}\mathbf{E}_0\right) = \frac{1}{c}(-\omega \mathbf{B}_0) = -\frac{\omega}{c}\mathbf{B}_0 &= -\frac{\omega}{c^2}\mathbf{E}_*. \end{aligned}$$

This can be used to find solutions with longitudinal or transverse magnetic multipoles at lowest order from those given in the previous two sections.

4 Other Approaches

It is also possible to express $\nabla \times (\nabla \times \mathbf{E}_0) = \nabla(\nabla \cdot \mathbf{E}_0) - \nabla^2 \mathbf{E}_0$ where $\nabla^2 \mathbf{E}_0$ is the vector Laplacian that must be chosen appropriately for the coordinate system being used. This has a simpler form than the double curl, so assuming that $\nabla \cdot \mathbf{E}_0 = 0$ by one of Maxwell's equations, it gets to the second-order recurrence relations more quickly. The expression $\nabla^2 \mathbf{E}_0 = -\frac{\omega^2}{c^2}\mathbf{E}_0$ is known as the Helmholtz equation. The downside of this approach is that the first-order recurrence relation is not recovered, because it comes from the assumption that $\nabla \cdot \mathbf{E}_0 = 0$, so it must be added manually by substituting the series for $\mathbf{E}_0$ into that equation.

5 Implementation Notes

It is possible to set $\omega = 0$ in the non-angular-dependent formula to get an entirely electrostatic solution. For the angularly-dependent case, multiply both fields by ω to eliminate the factor of $\frac{1}{\omega}$ on the $\mathbf{B}$ field before taking the limit $\omega \to 0$. The static solution is valid for either $\mathbf{E}$ or $\mathbf{B}$.

These formulae work well when the repeated derivatives of the on-axis function(s) such as $f^{(n)}$ are easy to calculate. One example is when the entrance and exit ramps of a field area are constructed from $\tanh(z/l)$ for some fringe field length l. As $\frac{\mathrm{d}}{\mathrm{d}z}\tanh z = 1 - \tanh^2 z$, the n^{th} derivative is an $(n+1)^{\text{th}}$ order polynomial in $\tanh z$. For a field of length L, the exit ramp looks like $\tanh((L-z)/l)$, which has similar repeated derivatives except the odd order ones have changed sign. This corresponds to changing the sign of the h terms in the no angular dependency case.

Once an array of repeated derivatives $[f, f', f'' \cdots f^{(N)}]$ has been calculated, the core operation is applying the operator $\frac{\omega^2}{c^2} + \frac{\partial^2}{\partial z^2}$ to it repeatedly. This is easy, since scalar multiplication works as usual and differentiation shifts the array to the left. Additionally, the second entry remains the derivative of the first, which is useful since both f_{2k} and f'_{2k} are required after each application of the second-order operator.

References

[1] S. J. Brooks, "Off-axis magnetic fields extrapolated from on-axis multipoles", BNL, Upton, NY, USA, Rep. BNL-223622-2022-TECH, 2013. doi:10.2172/1895080