# Integrated Fields of Permanent Magnet Dipole Trims

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To adjust the integrated dipole field of a magnet (through the longitudinal $z$ axis), pairs of small permanent magnets may be added at the end, symmetrically above and below the axis. If these are sufficiently small, they may be approximated by magnetic dipole point sources. The field of a single dipole source located at the origin is

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3} = \frac{\mu_0}{4\pi}\frac{3\mathbf{x}(\mathbf{x}\cdot\mathbf{m}) - \mathbf{m}|\mathbf{x}|^2}{|\mathbf{x}|^5},$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ and $\mathbf{m}$ is the magnetic dipole moment, defined as

$$\mathbf{m} = V\mathbf{M} = V\frac{\mathbf{B}_r}{\mu_0}$$

for a permanent magnet of volume $V$ with residual field vector $\mathbf{B}_r$.

Suppose the source is a distance $y$ vertically below the axis, then setting $\mathbf{x} = (0, y, z)$ will give the fields on-axis as a function of $z$. To get a vertical dipole field, assume the source is magnetised in the $y$ direction, so that $\mathbf{m} = (0, VB_{ry}/\mu_0, 0)$. The field is now given by

$$\begin{aligned}\mathbf{B}(0, y, z) &= \frac{\mu_0}{4\pi}\frac{3(0, y, z)(yVB_{ry}/\mu_0) - (0, VB_{ry}/\mu_0, 0)(y^2 + z^2)}{(y^2 + z^2)^{5/2}} \\ &= \frac{VB_{ry}}{4\pi}\frac{3(0, y, z)y - (0, 1, 0)(y^2 + z^2)}{(y^2 + z^2)^{5/2}}.\end{aligned}$$

The unwanted longitudinal field component can be eliminated by adding a second identical source a distance $y$ above the axis, producing a combined field from the two sources of:

$$\mathbf{B}^{\pm y}(z) = \mathbf{B}(0, y, z) + \mathbf{B}(0, -y, z) = \frac{VB_{ry}}{2\pi}\frac{3(0, y^2, 0) - (0, 1, 0)(y^2 + z^2)}{(y^2 + z^2)^{5/2}},$$

which has only a vertical component

$$B_y^{\pm y}(z) = \frac{VB_{ry}}{2\pi}\frac{2y^2 - z^2}{(y^2 + z^2)^{5/2}}.$$

The maximum field is achieved at $z = 0$:

$$B_y^{\pm y}(0) = \frac{VB_{ry}}{2\pi}\frac{2}{|y|^3},$$

although note at far distances $|z| > \sqrt{2}|y|$, the field actually becomes negative but with a much smaller magnitude. The integrated field is

$$\begin{aligned}
\int_{-\infty}^{\infty} B_y^{\pm y}(z)\,\mathrm{d}z &= \frac{VB_{ry}}{2\pi}\int_{-\infty}^{\infty}\frac{2y^2-z^2}{(y^2+z^2)^{5/2}}\,\mathrm{d}z \\
&= \frac{VB_{ry}}{2\pi}\left[\frac{z(2y^2+z^2)}{y^2(y^2+z^2)^{3/2}}\right]_{-\infty}^{z=\infty} \\
&= \frac{VB_{ry}}{2\pi}\frac{2}{y^2}.
\end{aligned}$$

This leads to the relation

$$\int_{-\infty}^{\infty} B_y^{\pm y}(z)\,\mathrm{d}z = B_y^{\pm y}(0)|y|,$$

in other words, integrated field is just peak field multiplied by the vertical distance to the magnets. This is useful empirically because it is often quicker to measure a peak field with a Hall probe than an integrated field through an infinitely-long axis.